

# Update Strength in EDAs and ACO: How to Avoid Genetic Drift

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## Abstract

We provide a rigorous runtime analysis concerning the update strength, a vital parameter in probabilistic model-building GAs such as the step size  $1/K$  in the compact Genetic Algorithm (cGA) and the evaporation factor  $\rho$  in ACO. While a large update strength is desirable for exploitation, there is a general trade-off: too strong updates can lead to genetic drift and poor performance. We demonstrate this trade-off for the cGA and a simple MMAS ACO algorithm on the OneMax function. More precisely, we obtain lower bounds on the expected runtime of  $\Omega(K\sqrt{n} + n \log n)$  and  $\Omega(\sqrt{n}/\rho + n \log n)$ , respectively, showing that the update strength should be limited to  $1/K, \rho = O(1/(\sqrt{n} \log n))$ . In fact, choosing  $1/K, \rho \sim 1/(\sqrt{n} \log n)$  both algorithms efficiently optimize OneMax in expected time  $O(n \log n)$ . Our analyses provide new insights into the stochastic behavior of probabilistic model-building GAs and propose new guidelines for setting the update strength in global optimization.

## 1 Introduction

The term *probabilistic model-building GA* describes a class of algorithms that construct a probabilistic model which is used to generate new search points. The model is adapted using information about previous search points. Both estimation-of-distribution algorithms (EDAs) and swarm intelligence algorithms including ant colony optimizers (ACO) and particle swarm optimizers (PSO) fall into this class. These algorithms generally behave differently from evolutionary algorithms where a population of search points fully describes the current state of the algorithm.

EDAs like the compact Genetic Algorithm (cGA) and many ACO algorithms update their probabilistic models by sampling new solutions and then updating the model according to information about good solutions found. In this work we focus on binary search spaces and simple univariate probabilistic models, that is, for each bit there is a value  $p_i$  that determines the probability of setting the  $i$ -th bit to 1 in a newly created solution.

The compact Genetic Algorithm was introduced by Harik, Lobo and Goldberg [12]. In brief, it simulates the behavior of a Genetic Algorithm with population size  $K$  in a more compact fashion. In each iteration two solutions are generated, and if they differ in fitness,  $p_i$  is updated by  $\pm 1/K$  in the direction of the fitter individual. Here  $1/K$  reflects the strength of the update of the probabilistic model. Simple ACO algorithms based on the Max-Min Ant System (MMAS) [23], using the iteration-best update rule, behave similarly: they generate a number  $\lambda$  of solutions and reinforce the best solution amongst these by increasing values  $p_i$ , here called *pheromones*, according to  $(1 - \rho)p_i + \rho$  if the best solution had bit  $i$  set to 1, and  $(1 - \rho)p_i$  otherwise. Here the parameter  $0 < \rho < 1$  is called *evaporation factor*; it plays a similar role to the update strength  $1/K$  for cGA.

Neumann, Sudholt, and Witt [18] showed that  $\lambda = 2$  ants suffice to optimize the function  $\text{OneMax}(x) := \sum_{i=1}^n x_i$ , a simple hill-climbing task, in expected time  $O(n \log n)$  if the update strength is chosen small enough,  $\rho \leq 1/(c\sqrt{n} \log n)$  for a suitably large constant  $c > 0$ . If  $\rho$  is chosen unreasonably large,  $\rho \geq c'/(\log n)$  for some  $c' > 0$ , the algorithm shows a chaotic behavior and needs exponential time even on this very simple function. In a more general sense, this result suggests that for global optimization such high update strengths should be avoided for any problem, unless the problem contains many global optima.

However, these results leave open a wide gap of parameter values between  $\sim 1/(\log n)$  and  $\sim 1/(\sqrt{n} \log n)$ , for which no results are available. This leaves open the question of which update strengths are optimal, and for which values performance degrades. Understanding the working principles of the underlying probabilistic model remains an important open problem for both cGA and ACO algorithms. This is evident from the lack of reasonable lower bounds. To date, the best known direct lower bound for MMAS algorithms for reasonable parameter choices is  $\Omega((\log n)/\rho - \log n)$  [17, Theorem 5]. The best known lower bound for cGA is  $\Omega(K\sqrt{n})$  [7]. There are more general bounds from black-box complexity theory [6, 8], showing that the expected runtime of comparison-based algorithms such as MMAS must be  $\Omega(n)$  on ONEMAX. However, these black-box bounds do not yield direct insight into the stochastic behavior of the algorithms and do not shed light on the dependency of the algorithms' performance on the update strength.

In this paper, we study 2-MMAS<sub>ib</sub> and cGA with a much more detailed analysis that provides such insights through rigorous runtime analysis. We prove lower bounds of  $\Omega(K\sqrt{n} + n \log n)$  and  $\Omega(1/\rho \cdot \sqrt{n} + n \log n)$ . The terms  $K\sqrt{n}$  and  $1/\rho \cdot \sqrt{n}$  indicate that the runtime decreases when the update strength  $1/K$  or  $\rho$  is increased. However, the added terms  $+ n \log n$  set a limit: there is no asymptotic decrease and hence no benefit for choosing update strengths  $1/K$  or  $\rho$  growing faster than  $1/(\sqrt{n} \log n)$ . The reason is that in this regime both algorithms suffer from genetic drift that leads to incorrect decisions being made. Correcting these incorrect decisions requires time  $\Omega(n \log n)$ . These lower bounds hold in expectation and with high probability; hence, they accurately reflect the algorithms' typical performance.

We further show that these bounds are tight for  $1/K, \rho \leq 1/(c\sqrt{n} \log n)$ . In this parameter regime the impact of genetic drift is bounded and hence these parameter

choices provably lead to the best asymptotic performance on OneMax for arbitrary problem sizes  $n$ .

The lower bounds formally apply to OneMax, but can be regarded as general limitations for global optimization on functions with a small number of optima. Among all functions with a unique global optimum, the function OneMax is provably the easiest function for certain evolutionary algorithms (see [5] for a proof for the (1+1) EA and [24, 25] for extensions to populations), and similar results were shown for the cGA on linear functions by Droste [7]. We believe that the lower bounds give general performance limits for all functions with a unique global optimum (however, new arguments will be required to show this formally).

From a technical point of view, our work uses a novel approach: using a second-order potential function to approximate the distribution of hitting times for a random walk that underlies changes in the probabilistic model. We are confident that this approach will find application in other stochastic processes.

Finally, by pointing out similarities between cGA and 2-MMAS<sub>ib</sub>, using the same analytical framework to understand changes in the probabilistic model, we make a step towards a unified theory of probabilistic model-building GAs.

This report is structured as follows. Section 2 introduces the algorithms and Section 3 presents important analytical concepts. Section 4 proves efficient upper bounds for small update strengths, whereas Section 5 deals with the lower bounds for large update strengths. We finish with some conclusions.

## 2 Preliminaries

Our presentation of cGA follows Droste [7]; see also Friedrich, Kötzing, Krejca, and Sutton [10]. The parameter  $1/K$  is called update strength (classically,  $K$  is called population size) and the  $p_{i,t}$  are called marginal probabilities. Pseudocode of cGA is shown in Algorithm 1. The simple MMAS algorithm 2-MMAS<sub>ib</sub>, analyzed before in [18]<sup>1</sup>, is shown in Algorithm 2. Note that the two algorithms only differ in the update mechanism. In the context of ACO,  $p_{i,t}$  are usually called pheromone values, however we also refer to them as marginal probabilities to unify our approach to both algorithms.

We note that the marginal probabilities for both algorithms are restricted to the interval  $[1/n, 1 - 1/n]$ . These bounds are used such that the algorithms always show a finite expected optimization time, as otherwise certain bits can be irreversibly fixed to 0 or 1. Our results also apply to algorithms without these borders: our analysis can be easily adapted to show that when the optimum is found efficiently in the presence of borders, it is found with high probability when borders are removed, and when the algorithm is inefficient, many bits are fixed opposite to the optimum.

There are intriguing similarities in the definition of cGA and 2-MMAS<sub>ib</sub>, despite these two algorithms coming from quite different strands from the EC community. As

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<sup>1</sup>The 2-MMAS<sub>ib</sub> in [18] used a randomized tie-breaking rule, which we replaced by a deterministic one here. This does not affect the stochastic behavior on ONEMAX but eases the analysis.

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**Algorithm 1:** Compact Genetic Algorithm (cGA)

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 $t \leftarrow 0$   $p_{1,t} \leftarrow p_{2,t} \leftarrow \dots \leftarrow p_{n,t} \leftarrow 1/2$  while termination criterion not met do  
  for  $i \in \{1, \dots, n\}$  do  
     $x_i \leftarrow 1$  with prob.  $p_{i,t}$ ,  $x_i \leftarrow 0$  with prob.  $1 - p_{i,t}$   
  for  $i \in \{1, \dots, n\}$  do  
     $y_i \leftarrow 1$  with prob.  $p_{i,t}$ ,  $y_i \leftarrow 0$  with prob.  $1 - p_{i,t}$   
  if  $f(x) < f(y)$  then swap  $x$  and  $y$  for  $i \in \{1, \dots, n\}$  do  
    if  $x_i > y_i$  then  $p_{i,t+1} \leftarrow p_{i,t} + 1/K$  if  $x_i < y_i$  then  $p_{i,t+1} \leftarrow p_{i,t} - 1/K$  if  
     $x_i = y_i$  then  $p_{i,t+1} \leftarrow p_{i,t}$  Restrict  $p_{i,t+1}$  to be within  $[1/n, 1 - 1/n]$   
   $t \leftarrow t + 1$ 
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**Algorithm 2:** 2-MMAS<sub>ib</sub>

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 $t \leftarrow 0$   $p_{1,t} \leftarrow p_{2,t} \leftarrow \dots \leftarrow p_{n,t} \leftarrow 1/2$  while termination criterion not met do  
  for  $i \in \{1, \dots, n\}$  do  
     $x_i \leftarrow 1$  with prob.  $p_{i,t}$ ,  $x_i \leftarrow 0$  with prob.  $1 - p_{i,t}$   
  for  $i \in \{1, \dots, n\}$  do  
     $y_i \leftarrow 1$  with prob.  $p_{i,t}$ ,  $y_i \leftarrow 0$  with prob.  $1 - p_{i,t}$   
  if  $f(x) < f(y)$  then swap  $x$  and  $y$  for  $i \in \{1, \dots, n\}$  do  
    if  $x_i \geq y_i$  then  $p_{i,t+1} \leftarrow (1 - \rho)p_{i,t} + \rho$  if  $x_i < y_i$  then  $p_{i,t+1} \leftarrow (1 - \rho)p_{i,t}$   
    Restrict  $p_{i,t+1}$  to be within  $[1/n, 1 - 1/n]$   
   $t \leftarrow t + 1$ 
```

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said, they only differ in the update mechanism: cGA uses a symmetrical update rule with  $1/K$  as the amount of change and changes a marginal probability if and only if both offspring differ in the corresponding bit value. 2-MMAS<sub>ib</sub> will always change a marginal probability in either positive or negative direction by a value dependent on its current state; however, the maximum absolute change will always be at most  $\rho$ . We are not the first to point out these similarities (e. g., see the survey by Hauschild and Pelikan [13], who embrace both algorithms under the umbrella of EDAs). However, our analyses will reveal the surprising insight that both cGA and 2-MMAS<sub>ib</sub> have the same runtime behavior as well as the same optimal parameter set on ONEMAX and can be analyzed with almost the same techniques.

### 3 On the Dynamics of the Probabilistic Model

We first elaborate on the stochastic processes underlying the probabilistic model in both algorithms. These insights will then be used to prove upper runtime bounds for small update strengths in Section 4 and lower runtime bounds for large update strengths in Section 5.

We fix an arbitrary bit  $i$  and  $p_{i,t}$ , its marginal probability at time  $t$ . Note that  $p_{i,t}$  is a random variable, and so is its random change  $\Delta_t := p_{i,t+1} - p_{i,t}$  in one step. This change

depends on whether the value of bit  $i$  matters for the decision whether to update with respect to the first bit string  $x$  sampled in iteration  $t$  (using  $p_{\cdot,t}$  as sampling distribution) or the second one  $y$  (cf. also [18]). More precisely, we inspect  $D_t := |x| - |x_i| - (|y| - |y_i|)$ , which is the change of ONEMAX-value at bits other than  $i$ .

We assume  $p_{i,t}$  to be bounded away from the borders such that  $\Delta_t$  is not affected by the borders. Then we get for cGA:

- If  $|D_t| \geq 2$ , then bit  $i$  does not affect the decision whether to update with respect to  $x$  or  $y$ . For  $\Delta_t > 0$  it is necessary that bit  $i$  is sampled differently. Hence, the  $p_{i,t}$ -value increases and decreases by  $1/K$  with equal probability  $p_{i,t}(1 - p_{i,t})$ ; with the remaining probability  $p_{i,t+1} = p_{i,t}$ . The change in this case is defined by  $\Delta_t = F_t$  where

$$F_t := \begin{cases} +1/K & \text{with probability } p_{i,t}(1 - p_{i,t}), \\ -1/K & \text{with probability } p_{i,t}(1 - p_{i,t}), \\ 0 & \text{with the remaining probability.} \end{cases}$$

We call a step where  $|D_t| \geq 2$  a *random-walk step* (*rw-step*) since the process in such a step is a fair random walk (with self-loops) as  $E(\Delta_t \mid p_{i,t}) = E(F_t \mid p_{i,t}) = 0$ .

If  $D_t = 1$  then  $|x_{t+1}| \geq |y_{t+1}|$  such that  $x_{t+1}$  and  $y_{t+1}$  are never swapped in line 8 of cGA. Hence, the same argumentation as in the previous case applies and the process performs an rw-step as well.

- If  $D_t = -1$  then  $x_{t+1}$  and  $y_{t+1}$  are swapped unless bit  $i$  is sampled to 1 in  $x_{t+1}$  and to 0 in  $y_{t+1}$ . Hence, both events of sampling bit  $i$  differently increase the  $p_{i,t}$ -value. We have  $\Delta_t = 1/K$  with probability  $2p_{i,t}(1 - p_{i,t})$  and  $\Delta_t = 0$  otherwise.

If  $D_t = 0$  then as in the case  $D_t = -1$  both events of sampling bit  $i$  differently increase the  $p_{i,t}$ -value. Hence, we again have  $\Delta_t = 1/K$  with probability  $2p_{i,t}(1 - p_{i,t})$  and  $\Delta_t = 0$  otherwise. Let  $B_t$  be a random variable such that

$$B_t := \begin{cases} +1/K & \text{with probability } 2p_{i,t}(1 - p_{i,t}), \\ 0 & \text{with the remaining probability.} \end{cases}$$

Hence, in the cases  $D_t = -1$  and  $D_t = 0$  we get  $\Delta_t = B_t$ . We call such a step a *biased step* (*b-step*) since  $E(\Delta_t \mid p_{i,t}) = E(B_t \mid p_{i,t}) = 2p_{i,t}(1 - p_{i,t})/K > 0$  here.

Whether a step is an rw-step or b-step for bit  $i$  depends only on circumstances being external to the bit (and independent of it). Let  $R_t$  be the event that  $D_t = 1$  or  $|D_t| \geq 2$ . We get the equality

$$\Delta_t = F_t \cdot \mathbb{P}[R_t] + B_t \cdot (1 - \mathbb{P}[R_t]), \quad (1)$$

which we denote as *superposition*. Informally, the change of  $p_{i,t}$ -value is a superposition of a fair (unbiased) random walk and biased steps. The fair random walk reflects the *genetic drift* underlying the process, i.e. the variance in the process may lead the algorithm to move in a random direction. In contrast, the biased steps reflect steps where the

algorithm *learns* about which bit value leads to a better fitness at the considered bit position. We remark that the superposition of two different behaviors as formulated here is related to the approach taken in [2], where an EDA called UMDA was decomposed into a derandomized, deterministic EDA and a stochastic component modeling genetic drift.

For 2-MMAS<sub>ib</sub>, structurally this kind of superposition holds as well, however, the underlying random variables look somewhat different. We have:

- If  $|D_t| \geq 2$  or  $D_t = 1$ , then the considered bit does not affect the choice whether to update with respect to  $x$  or  $y$ . Hence, the marginal probability of the considered bit increases with probability  $p_{i,t}$  and decreases with probability  $1 - p_{i,t}$ .

We get  $\Delta_t = p_{i,t+1} - p_{i,t} = F_t$  in this case, where  $F_t$  is a random variable such that

$$F_t := \begin{cases} \rho \cdot (1 - p_{i,t}) & \text{with probability } p_{i,t}, \\ -\rho \cdot p_{i,t} & \text{with probability } 1 - p_{i,t}. \end{cases}$$

We call such a step an rw-step in analogy to cGA as here  $E(\Delta_t | p_{i,t}) = E(F_t | p_{i,t}) = 0$ .

- If  $D_t = 0$  or  $D_t = -1$  then the marginal probability can only decrease if both offspring sample a 0 at bit  $i$ ; otherwise it will increase. The difference  $\Delta_t$  is a random variable

$$B_t := \begin{cases} \rho \cdot (1 - p_{i,t}) & \text{with probability } 1 - (1 - p_{i,t})^2, \\ -\rho \cdot p_{i,t} & \text{with probability } (1 - p_{i,t})^2. \end{cases}$$

The step is called a biased step (b-step) as  $E(\Delta_t | p_{i,t}) = E(B_t | p_{i,t}) = \rho p_{i,t}(1 - p_{i,t}) > 0$ .

Altogether, the superposition for 2-MMAS<sub>ib</sub> is also given by (1), with the modified meaning of  $B_t$  and  $F_t$ .

The strength of the update plays a key role here: if the update is too strong, large steps are made during updates, and genetic drift through rw-steps may overwhelm the probabilistic model, leading to “wrong” decisions being made in individual bits. On the other hand, small updates imply that rw-steps have a bounded impact, and the algorithm receives more time to learn optimal bit values in b-steps. We will formalize these insights in the following sections en route to proving rigorous upper and lower runtime bounds. Informally, one main challenge is to understand the stochastic process induced by the mixture of b- and rw-steps.

## 4 Small Update Strengths are Efficient

We first show that small update strengths are efficient. This has been shown for 2-MMAS<sub>ib</sub> in [18].

**Theorem 1** ([18]). *If  $\rho \leq 1/(cn^{1/2} \log n)$  for a sufficiently large constant  $c > 0$  and  $\rho \geq 1/\text{poly}(n)$  then 2-MMAS<sub>ib</sub> optimizes ONEMAX in expected time  $O(\sqrt{n}/\rho)$ .*

*For  $\rho = 1/(cn^{1/2} \log n)$  the runtime bound is  $O(n \log n)$ .*

Here we exploit the similarities between both algorithms to prove an analogous result for cGA.

**Theorem 2.** *The expected optimization time of cGA on ONEMAX with  $K \geq c\sqrt{n} \log n$  for a sufficiently large  $c > 0$  and  $K = \text{poly}(n)$  is  $O(\sqrt{n}K)$ . This is  $O(n \log n)$  for  $K = c\sqrt{n} \log n$ .*

The analysis follows the approach for 2-MMAS<sub>ib</sub> in [18], adapted to the different update rule, and using modern tools like *variable drift analysis* [14]. The main idea is that marginal probabilities are likely to increase from their initial values of  $1/2$ . If the update strength is chosen small enough, the effect of genetic drift (as present in rw-steps) is bounded such that with high probability all bits never reach marginal probabilities below  $1/3$ . Under this condition, we show that the marginal probabilities have a tendency (stochastic drift) to move to their upper borders, such that then the optimum is found with good probability.

The following lemma uses considerations and notation from Section 3 to establish a *stochastic drift*, i.e. a positive trend towards optimal bit values, for cGA. We use the same notation as in Section 3.

**Lemma 3.** *If  $1/n + 1/K \leq p_{i,t} \leq 1 - 1/n - 1/K$  then*

$$\mathbb{E}(\Delta_t \mid p_{i,t}) \geq \frac{2}{11} \frac{p_{i,t}(1 - p_{i,t})}{K} \left( \sum_{j \neq i} p_{j,t}(1 - p_{j,t}) \right)^{-1/2}.$$

**Proof.** The assumptions on  $p_{i,t}$  assure that  $p_{i,t+1}$  is not affected by the borders  $1/n$  and  $1 - 1/n$ . Then the expected change is given by the expectation of the superposition (1):

$$\mathbb{E}(\Delta_t \mid p_{i,t}) = \mathbb{E}(F_t \mid p_{i,t}) \cdot \mathbb{P}[R_t] + \mathbb{E}(B_t \mid p_{i,t}) \cdot (1 - \mathbb{P}[R_t]).$$

From Section 3 we know  $\mathbb{E}(F_t \mid p_{i,t}) = 0$  and  $\mathbb{E}(B_t \mid p_{i,t}) = 2p_{i,t}(1 - p_{i,t})/K$ . Further,

$$1 - \mathbb{P}[R_t] \geq \mathbb{P}[D_t = 0] \geq \frac{1}{11} \left( \sum_{j \neq i} p_{j,t}(1 - p_{j,t}) \right)^{-1/2},$$

where the last inequality was shown in [18, proof of Lemma 1]. Here we exploit that cGA and 2-MMAS<sub>ib</sub> use the same construction procedure. Together this proves the claim.  $\square$

Note that the term  $\left( \sum_{j \neq i} p_{j,t}(1 - p_{j,t}) \right)^{1/2}$  reflects the standard deviation of the sampling distribution on all bits  $j \neq i$ .

Lemma 3 indicates that the drift increases with the update strength  $1/K$ . However, a too large value for  $1/K$  also increases genetic drift. The following lemma shows that, if  $1/K$  is not too large, this positive drift implies that the marginal probabilities will generally move to higher values and are unlikely to decrease by a large distance.

**Lemma 4.** *Let  $0 < \alpha < \beta < 1$  be two constants. For each constant  $\gamma > 0$  there exists a constant  $c_\gamma > 0$  (possibly depending on  $\alpha, \beta$ , and  $\gamma$ ) such that for a specific bit the following holds. If the bit has marginal probability at least  $\beta$  and  $K \geq c_\gamma \sqrt{n} \log n$  then the probability that during the following  $n^\gamma$  steps the marginal probability decreases below  $\alpha$  is at most  $n^{-\gamma}$ .*

**Proof.** The proof is essentially the same as the proof of Lemma 3 in [18], using  $1/K$  instead of  $\rho$  and drift bounds from Lemma 3.  $\square$

With these lemmas, we now prove the main statement of this section.

**Proof of Theorem 2.** We assume in the following that  $1/K$  is a multiple of  $1/2 - 1/n$ , implying that marginal probabilities are restricted to  $\{1/n, 1/n + 1/K, \dots, 1/2, \dots, 1 - 1/n - 1/K, 1 - 1/n\}$ .

Following [18, Theorem 3] we show that, starting with a setting where all probabilities are at least  $1/2$  simultaneously, with probability  $\Omega(1)$  after  $O(\sqrt{n}K)$  iterations either the global optimum has been found or at least one probability has dropped below  $1/3$ . In the first case we speak of a success and in the latter case of a failure. The expected time until either a success or a failure happens is then  $O(\sqrt{n}K)$ .

Now choose a constant  $\gamma > 0$  such that  $n^\gamma \geq Kn^3$ . According to Lemma 4 applied with  $\alpha := 1/3$  and  $\beta := 1/2$ , the probability of a failure in  $n^\gamma$  iterations is at most  $n^{-\gamma}$ , provided the constant  $c$  in the condition  $K \geq c\sqrt{n} \log n$  is large enough. In case of a failure we wait until the probabilities simultaneously reach values at least  $1/2$  again and then we repeat the arguments from the preceding paragraph. It is easy to show (cf. Lemma 2 in [18]) that the expected time for one probability to reach the upper border is always bounded by  $O(n^{3/2}K)$ , regardless of the initial probabilities. By standard arguments on independent phases, the expected time until *all* probabilities have reached their upper border at least once is  $O(n^{3/2}K \log n)$ . Once a bit reaches the upper border, we apply Lemma 4 again with  $\alpha := 1/2$  and  $\beta := 2/3$  to show that the probability of a marginal probability decreasing below  $1/2$  in time  $n^\gamma$  is at most  $n^{-\gamma}$  (again, for large enough  $c$ ). The probability that there is a bit for which this happens is at most  $n^{-\gamma+1}$  by the union bound. If this does not happen, all bits attain value at least  $1/2$  simultaneously, and we apply our above arguments again.

As the probability of a failure is at most  $n^{-\gamma+1}$ , the expected number of restarts is  $O(n^{-\gamma+1})$  and considering the expected time until all bits recover to values at least  $1/2$  only leads to an additional term of  $n^{-\gamma+1} \cdot O((n^{3/2} \log n)K) \leq o(1)$  (as  $n^{-\gamma} \leq n^{-3}/K$ ) in the expectation.

We only need to show that after  $O(\sqrt{n}K)$  iterations without failure the probability of having found the global optimum is  $\Omega(1)$ . To this end, we consider a simple potential function that takes into account marginal probabilities for all bits. An important property of the potential is that once the potential has decreased to some constant value, the probability of generating the global optimum is constant.

Let  $p_1, \dots, p_n$  be the current marginal probabilities and  $q_i := 1 - 1/n - p_i$  for all  $i$ . Define the potential function  $\varphi := \sum_{i=1}^n q_i$ , which measures the distance to an ideal



setting where all probabilities attain their maximum  $1 - 1/n$ . Let  $q'_i$  be the  $q_i$ -value in the next iteration and  $p'_i = 1 - q'_i$ . We estimate the expectation of  $\varphi' := \sum_{i=1}^n q'_i$  and distinguish between two cases. If  $p_i \leq 1 - 1/n - 1/K$ , by Lemma 3

$$\mathbb{E}(q'_i | q_i) \leq q_i - \frac{p_i(1 - p_i)}{K} \cdot \frac{2}{11} \cdot \left( \sum_{j \neq i} p_j(1 - p_j) \right)^{-1/2}.$$

We bound  $p_i(1 - p_i)$  from below using  $p_i \geq 1/3$  and  $1 - p_i \leq 1 - 1/n - p_i = q_i$  and the sum from above using

$$\sum_{j \neq i} p_j(1 - p_j) \leq \sum_{j=1}^n (1 - p_j) = \sum_{j=1}^n (q_j + 1/n) = 1 + \varphi.$$

Then

$$\begin{aligned} \mathbb{E}(q'_i | q_i) &\leq q_i - \frac{q_i}{K} \cdot \frac{2}{33} \cdot \left( \frac{1}{1 + \varphi} \right)^{1/2} \\ &\leq q_i \left( 1 - \frac{2}{33K} \cdot \frac{1}{1 + \varphi^{1/2}} \right). \end{aligned}$$

If  $p_i > 1 - 1/n - 1/K$ , then  $p_i = 1 - 1/n$  (as  $1/K$  is a multiple of  $1/2 - 1/n$ ) and  $p_i$  can only decrease. A decrease by  $1/K$  happens with probability  $1/n$ , thus

$$\mathbb{E}(q'_i | q_i) \leq q_i + \frac{1}{nK}.$$

To ease the notation we assume w.l.o.g. that the bits are numbered according to decreasing probabilities, i.e., increasing  $q$ -values. Let  $m \in \mathbb{N}_0$  be the largest index such that  $p_m = 1 - 1/n$ . It follows

$$\sum_{i=1}^m \mathbb{E}(q'_i | q_i) \leq \sum_{i=1}^m q_i + \frac{m}{nK} \leq \sum_{i=1}^m q_i + \frac{1}{K}.$$

Putting everything together and using  $\sum_{i=1}^m q_i = \frac{m}{n} \leq 1$ ,

$$\begin{aligned} \mathbb{E}(\varphi' | \varphi) &= \sum_{i=1}^m \mathbb{E}(q'_i | q_i) + \sum_{i=m+1}^n \mathbb{E}(q'_i | q_i) \\ &\leq \sum_{i=1}^m q_i + \frac{1}{K} + \sum_{i=m+1}^n q_i \left( 1 - \frac{2}{33K} \cdot \frac{1}{1 + \varphi^{1/2}} \right) \\ &\leq 1 + \frac{1}{K} + (\varphi - 1) \left( 1 - \frac{2}{33K} \cdot \frac{1}{1 + \varphi^{1/2}} \right) \\ &\leq \varphi \left( 1 - \frac{2}{33K} \cdot \frac{1}{1 + \varphi^{1/2}} \right) + \frac{3}{K} \end{aligned}$$

where in the last line we used  $\frac{2}{33K} \cdot \frac{1}{1+\varphi^{1/2}} \leq \frac{2}{33K} \leq 2/K$ . For  $\varphi \geq 10000$  this can further be bounded using  $1 + \varphi^{1/2} \leq \varphi^{1/2}/100 + \varphi^{1/2} = 101/100 \cdot \varphi^{1/2}$ ,

$$\mathbb{E}(\varphi' \mid \varphi) \leq \varphi - \varphi^{1/2} \cdot \frac{101}{100} \cdot \frac{2}{33K} + \frac{3}{K} \leq \varphi - \varphi^{1/2} \cdot \frac{101}{3300K}$$

where in the last step we used  $\varphi^{1/2} \cdot \frac{101}{100} \cdot \frac{1}{33K} \geq \frac{101}{33K} \geq \frac{3}{K}$ , i. e., half of the negative term subsumes the  $+3/K$  term.

Now a straightforward generalization of variable drift theorem (given by Theorem 17 in the appendix), applied with a drift function of  $h(\varphi) := \varphi^{1/2} \cdot \frac{101}{3300K}$ , states that the expected time for  $\varphi$  to decrease from any initial value  $\varphi \leq n$  to a value  $\varphi \leq 10000$  is at most

$$\begin{aligned} & \frac{10000}{h(10000)} + \int_{10000}^n \frac{1}{h(\varphi)} d\varphi \\ &= O(K) + O(K) \cdot \int_{10000}^n \varphi^{-1/2} d\varphi = O(\sqrt{n}K). \end{aligned}$$

Consider an iteration where  $\varphi \leq 10000$ . The probability of creating ones on all bits simultaneously, given that all marginal probabilities are at least  $1/3$ , is minimal in the extreme setting where a maximal number of bits has marginal probabilities at  $1/3$  and all other bits, except at most one, have marginal probabilities at their upper border. Then the probability of creating the optimum in one step is at least  $(1 - \frac{1}{n})^{n-1} \cdot 3^{-\lceil \varphi \cdot 3/2 \rceil} = \Omega(1)$ . Hence a successful phase finds the optimum with probability  $\Omega(1)$ .  $\square$

## 5 Large Update Strengths Lead to Genetic Drift

The bound  $O(\sqrt{n}K)$  from Theorem 2 shows that larger update strengths (i. e., smaller  $K$ ) result in smaller bounds on the runtime. However, the theorem requires that  $K \geq c\sqrt{n} \log n$  so that the best possible choice results in  $O(n \log n)$  runtime. An obvious question to ask is whether this is only a weakness of the analysis or whether there is an intrinsic limit that prevents smaller choices of  $K$  from being efficient.

In this section, we will show that smaller choices of  $K$  (i. e., larger update strengths) cannot give runtimes of lower orders than  $n \log n$ . In a nutshell, even though larger update strengths support faster exploitation of correct decisions at single bits by quickly reinforcing promising bit values they also increase the risk of genetic drift reinforcing incorrectly made decisions at single bits too quickly. Then it typically happens that several marginal probabilities reach their lower border  $1/n$ , from which it (due to so-called coupon collector effects) takes  $\Omega(n \log n)$  steps to “unlearn” the wrong settings. The very same effect happens with 2-MMAS<sub>ib</sub> if its update strength  $\rho$  is chosen too large.

We now state the lower bounds we obtain for the two algorithms, see Theorems 5 and 6 below. Note that the statements are identical if we identify the update strength  $1/K$  of cGA with the update strength  $\rho$  of 2-MMAS<sub>ib</sub>. Also the proofs of these two

theorems will largely follow the same steps. Therefore, we describe the proof approach in detail with respect to cGA in Section 5.1. In Section 5.2, we describe the few places where slightly different arguments are needed to obtain the result for 2-MMAS<sub>ib</sub>.

**Theorem 5.** *The optimization time of cGA with  $K \leq \text{poly}(n)$  is  $\Omega(\sqrt{n}K + n \log n)$  with probability  $1 - \text{poly}(n) \cdot 2^{-\Omega(\min\{K, n^{1/2-o(1)}\})}$  and in expectation.*

**Theorem 6.** *The optimization time of 2-MMAS<sub>ib</sub> with  $1/\rho \leq \text{poly}(n)$  is  $\Omega(\sqrt{n}/\rho + n \log n)$  with probability  $1 - \text{poly}(n) \cdot 2^{-\Omega(\min\{1/\rho, n^{1/2-o(1)}\})}$  and in expectation.*

## 5.1 Proof of Lower Bound for cGA

We first describe at an intuitive level why large update strengths in cGA can be risky. In the upper bound from Theorem 2, we have shown that for sufficiently small update strengths, the positive stochastic drift by b-steps is strong enough such that even in the presence of rw-steps *all* bits never reach marginal probabilities below  $1/3$ , with high probability. Then no “incorrect” decision is made.

To prove Theorem 5, we show that with larger update strengths than  $1/(\sqrt{n} \log n)$  the effect of rw-steps is strong enough such that with high probability *some* bits will make an incorrect decision and reach the lower borders of marginal probabilities. We consider the hitting time for a marginal probability to reach the lower border  $1/n$  and analyze the distribution of this hitting time more closely.

To illustrate this setting, fix one bit and imagine that all steps were rw-steps (we will explain later how to handle b-steps), and that all rw-steps change the current value of the bit’s marginal probability (i. e., there are no self-loops). Then the process would be a fair random walk on  $\{0, 1/K, 2/K, \dots, (K-1)/K, 1\}$ , started at  $1/2$ . This fair random walk is well understood and it is well known that the hitting time is not sharply concentrated around the expectation. More precisely, there is still a polynomially in  $K$  small probability of hitting a border within at most  $O(K^2/\log K)$  steps and also of needing at least  $\Omega(K^2 \log K)$  steps. The underlying idea is that the Central Limit Theorem (CLT) approximates the progress within a given number of steps.

The real process is more complicated because of self-loops. Recall from the definition of  $F_t$  that the process only changes its current state by  $\pm 1/K$  with probability  $2p_{i,t}(1 - p_{i,t})$ , hence with probability  $1 - 2p_{i,t}(1 - p_{i,t})$  a self-loop occurs on this bit. The closer the process is to one of its borders  $\{1/n, 1 - 1/n\}$ , the larger the self-loop probability becomes and the more the random walk slows down. Hence the actual process is clearly slower in reaching a border since every looping step is just wasted. One might conjecture that the self-loops will asymptotically increase the expected hitting time. But interestingly, as we will show, the expected hitting time in the presence of self-loops is still of order  $\Theta(K^2)$ . Also the CLT (in a generalized form) is still applicable despite the self-loops, leading to a similar distribution as above.

The distribution of the hitting time of the random walk with self-loops will be analyzed in Lemma 7 below. In order to deal with self-loops, in its proof, we use a potential function mapping the actual process to a process on a scaled state space with nearly

position-independent variance. Unlike the typical applications of potential functions in drift analysis, the purpose of the potential function is not to establish a position-independent first-moment stochastic drift but a (nearly) position-independent variance, i.e., the potential function is designed to analyze a second moment. This argument seems to be new in the theory of drift analysis and may be of independent interest. The lemma also takes into account the b-steps in between rw-steps and shows how the rw-steps can still overwhelm the accumulated effect of b-steps if the latter are not too frequent.

**Lemma 7.** *Consider a bit of cGA on ONEMAX and let  $p_t$  be its marginal probability at time  $t$ . Let  $t_1, t_2, \dots$  be the times where cGA performs an rw-step (before hitting one of the borders  $1/n$  or  $1 - 1/n$ ) and let  $\Delta_i := p_{t_{i+1}} - p_{t_i}$ . For  $s \in \mathbb{R}$ , let  $T_s$  be the smallest  $t$  such that  $\text{sgn}(s) (\sum_{i=0}^t \Delta_i) \geq |s|$  holds or a border has been reached.*

*Choosing  $0 < \alpha < 1$ , where  $1/\alpha = o(K)$ , and  $-1 < s < 0$  constant, and assuming that at most  $|s|K/4$  of the steps until time  $t_{\alpha(sK)^2}$  are b-steps, we have*

$$\begin{aligned} \mathbb{P}[T_s \leq \alpha(sK)^2 \text{ or } p_t \text{ exceeds } 5/6 \text{ before } T_s] \\ \geq (1/2 - o(1)) \cdot \left( \frac{1}{13\sqrt{1/(|s|\alpha)}} - \frac{1}{(13\sqrt{1/(|s|\alpha)})^3} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{169}{2|s|\alpha}}. \end{aligned}$$

Moreover, for any  $\alpha > 0$  and  $s \in \mathbb{R}$ ,

$$\mathbb{P}[T_s \geq \alpha(sK)^2 \text{ or a border is reached until time } \alpha(sK)^2] \geq 1 - e^{-1/(4\alpha)}.$$

Informally, the lemma means that every deviation of the hitting time  $T_s$  by a constant factor from its expected value (which turns out as  $\Theta(s^2 K^2)$ ) still has constant probability, and even deviations by logarithmic factors have a polynomially small probability. We will mostly apply the lemma for  $\alpha < 1$ , especially  $\alpha \approx 1/\log n$ , to show that there are marginal probabilities that quickly approach the lower border; in fact, this effect implies the  $\log n$  term in the optimal update strength. Note that the second statement of the lemma also holds for  $\alpha \geq 1$ ; however, in this realm also Markov's inequality works. Then, by the inequality  $e^{-x} \leq 1 - x/2$  for  $x \leq 1$ , we get  $\mathbb{P}[T_s \geq \alpha s^2 K^2] \geq 1/(4\alpha)$ , which means that Markov's inequality for deviations above the expected value is asymptotically tight in this case.

To illustrate the main idea for the proof of Lemma 7, we ignore b-steps for a while and note that we are confronted with a fair random walk. However, the random walk is not longer homogeneous with respect to place as the self-loops slow the process down in the vicinity of a border. The random variables describing the change of position from time  $t$  to time  $t+1$  (formally,  $\Delta_t := p_{t+1} - p_t$ ) that are not identically distributed, other than in the classical fair random walk. In fact, the variance of  $\Delta_t$  becomes smaller the closer  $p_t$  is to one of the borders.

In more detail, the potential function used in Lemma 7 essentially uses the self-loop probabilities to construct extra distances to bridge. For instance, states with low self-loop probability (e.g.,  $1/2$ ), will have a potential that is only by  $\Theta(1)$  larger or smaller than the potential of its neighbors. On the other hand, states with a large self-loop

probability, say  $1/K$ , will have a potential that can differ by as much as  $2\sqrt{K}$  from the potential of its neighbors. Interestingly, this choice leads to variances of the one-step changes that are basically the same on the whole state space (very roughly, this is true since the squared change  $(2\sqrt{K})^2 = \Theta(K)$  is observed with probability  $\Theta(1/K)$ ). However, using the potential for this trick is at the expense of changing the support of the underlying random variables, which then will depend on the state. Nevertheless, as the support is not changed too much, the Central Limit Theorem (CLT) still applies and we can approximate the progress made within  $T$  steps by a normally distributed random variable. This approximation is made precise in the following lemma. See, Eq. (27.16) in [1].

**Lemma 8** (Weak CLT with Lyapunov condition). *Let  $X_1, \dots, X_n$  be a sequence of independent random variables, each with finite expected value  $\mu_i$  and variance  $\sigma_i^2$ . Define*

$$s_n^2 := \sum_{i=1}^n \sigma_i^2 \quad \text{and} \quad C_n := \frac{1}{s_n} \sum_{i=1}^n (X_i - \mu_i).$$

*If there exists  $\delta > 0$  such that*

$$\lim_{n \rightarrow \infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^n \mathbb{E}(|X_i - \mu_i|^{2+\delta}) = 0$$

*(assuming all the moments of order  $2+\delta$  to be defined), then  $C_n$  converges in distribution to a standard normally distributed random variable, more precisely for all  $x \in \mathbb{R}$*

$$\lim_{n \rightarrow \infty} \mathbb{P}[C_n \leq x] = \Phi(x),$$

*or, equivalently,*

$$|\mathbb{P}[C_n \leq x] - \Phi(x)| = o(1),$$

*where  $\Phi(x)$  is the cumulative distribution function of the standard normal distribution.*

We now turn to the formal proof.

**Proof of Lemma 7.** Throughout this proof, to ease notation we consider the scaled process on the state space  $S := \{0, 1, \dots, K\}$  obtained by multiplying all marginal probabilities by  $K$ ; the random variables  $X_t = Kp_t$  will live on this scaled space. Note that we also remove the borders ( $K/n$  and  $K - K/n$ ), which is possible as all considerations are stopped when such a border is reached. For the same reason, we only consider current states from  $\{1, \dots, K-1\}$  in the remainder of this proof.

The first hitting time  $T_s$  becomes only stochastically larger if we ignore all self-loops. Formally, recalling the trivial scaling of the state space, we consider the fair random walk where  $\mathbb{P}[X_{t_i+1} = j-1] = \mathbb{P}[X_{t_i+1} = j+1] = 1/2$  if  $X_{t_i} = j \in \{1, \dots, K-1\}$ . We write  $Y_t = \sum_{i=0}^{t-1} \Delta_{t_i}$ . Clearly,  $\Delta_i$  is uniform on  $\{-1, 1\}$ ,  $\mathbb{E}(\Delta_i \mid 0 < X_{t_i} < K) = 0$ ,  $\text{Var}(\Delta_i \mid 0 < X_{t_i} < K) = 1$  and  $Y_t$  is a sum of independent, identically distributed random variables. It is well known that  $(Y_t - \mathbb{E}(Y_t))/\sqrt{\text{Var}(Y_t)}$  converges in distribution

to a standard normally distributed random variable. However, we do not use this fact directly here. Instead, to bound the deviation from the expectation, we use a classical Hoeffding bound. We assume  $s \geq 0$  now and will see that the case  $s < 0$  can be handled symmetrically.

Theorem 1.11 in [4] yields, with  $c_i = 2$  as the size of the support of  $\Delta_i$ , that

$$\mathbb{P}[Y_{\alpha s^2 K^2} \geq sK] \leq e^{-(sK)^2/(4\alpha s^2 K^2)} = e^{-1/(4\alpha)}.$$

Moreover, according to Theorem 1.13 in [4], the bound also holds for all  $k \leq \alpha s^2 K^2$  together, more precisely,

$$\mathbb{P}[\exists k \leq \alpha s^2 K^2: Y_k \geq sK] \leq e^{-1/(4\alpha)}.$$

Symmetrically, we obtain

$$\mathbb{P}[\exists k \leq \alpha s^2 K^2: Y_k \leq -sK] \leq e^{-1/(4\alpha)}.$$

Hence, distance that is strictly smaller than  $sK$  is bridged through  $\alpha(sK)^2$  rw-steps (or the process reaches a border before) with probability at least  $1 - e^{-1/(4\alpha)}$ .

We are left with the first statement, where the stronger condition  $-1 < s < 0$  and  $|s| = \Omega(1)$  is made. Here we will essentially use an approximation of the accumulated state within  $\alpha s^2 K^2$  steps by the normal distribution, but have to be careful to take into account steps describing self-loops. To analyze the hitting time  $T_s$  for the  $X_{t_i}$ -process, we now define a potential function  $g: S \rightarrow \mathbb{R}$ . Unlike the typical applications of potential functions, the purpose of  $g$  is not to establish a position-independent first-moment drift (in fact, there is no drift within  $S$  since the original process is a martingale) but a (nearly) position-independent variance, i.e., the potential function is designed to analyze a second moment.

We proceed with the formal definition of the potential function, the analysis of its expected first-moment change and the corresponding variance, and a proof that the Lyapunov condition holds for the accumulated change within  $\alpha s^2 K^2$  steps. The potential function  $g$  is monotonically decreasing on  $\{1, \dots, K/2\}$  and centrally symmetric around  $K/2$ . We define it as follows: let  $g(K/2) = 0$  and for  $1 \leq i \leq K/2 - 1$ , let  $g(i) - g(i+1) = \sqrt{2K/(i+1)}$ ; finally, let  $g(K-i) = -g(i)$ . Inductively, we have

$$g(i) = -g(K-i) = \sum_{j=i}^{K/2-1} \sqrt{2K/(j+1)}$$

for  $1 \leq i \leq K/2$ . We note that  $g(0) = O(K)$ , more precisely it holds

$$g(0) = \sqrt{2K} \left( \sum_{j=1}^{K/2-1} \sqrt{1/(j+1)} \right) \leq \sqrt{2K} (2\sqrt{K/2}) = 2K.$$

More generally, for  $i < j \leq K/2$ , we get by the monotonicity of  $g$  that

$$g(i) - g(j) \leq g(0) - g(j-i) = \sqrt{2K} \sum_{k=1}^{j-i} \sqrt{1/k} \leq 2\sqrt{2K}(\sqrt{j-i}) \quad (2)$$

Informally, the potential function stretches the whole state space by a factor of at most 2 but adjacent states in the vicinity of borders can be by  $2\sqrt{K}$  apart in potential.

Let  $Y_t := g(X_t)$ . We consider the one-step differences  $\Psi_i := Y_{t_i+1} - Y_{t_i}$  at the times  $i$  where rw-steps occur, and we will show via the representation  $Y_{t_i} := \sum_{j=0}^{i-1} \Psi_j$  that  $Y_{t_i}$  approaches a normally distributed variable. Note that  $Y_{t_i}$  is not necessarily the same as  $g(X_{t_i}) - g(X_{t_0})$  since only the effect of rw-steps is covered by  $Y_{t_i}$ .

In the following, we assume  $1 \leq X_{t_i} \leq K/2$  and note that the case  $X_{t_i} > K/2$  can be handled symmetrically with respect to  $-\Psi_i$ . We claim that for all  $i \geq 0$

$$0 \leq \mathbb{E}(\Psi_i \mid X_{t_i}) \leq \sqrt{2/(X_{t_i}K)} \leq o(1), \quad (3)$$

$$1/4 \leq \text{Var}(\Psi_i \mid X_{t_i}), \quad (4)$$

where all  $O$ -notation is with respect to  $K$ .

The lower bound  $\mathbb{E}(\Psi_i \mid X_{t_i}) \geq 0$  is easy to see since  $X_{t_i}$  is a fair random walk and  $g(j-1) - g(j) \geq g(j) - g(j+1)$  holds for all  $j \leq K/2$ . To prove the upper bound, we note that  $X_{t_i+1} \in \{X_{t_i} - 1, X_{t_i}, X_{t_i} + 1\}$  so that

$$\mathbb{E}(\Psi_i \mid X_{t_i}) = \mathbb{P}[X_{t_i+1} < X_{t_i}](g(X_{t_i} - 1) - g(X_{t_i})) + \mathbb{P}[X_{t_i+1} > X_{t_i}](g(X_{t_i} + 1) - g(X_{t_i}))$$

Using the properties of rw-steps, we have that  $\mathbb{P}[Y_{t_i+1} \neq Y_{t_i}] = 2\frac{(K-X_{t_i})X_{t_i}}{K^2}$ . Moreover, on  $Y_{t_i+1} \neq Y_{t_i}$ ,  $Y_{t_i+1}$  takes each of the two values  $g(X_{t_i} - 1)$  and  $g(X_{t_i} + 1)$  with the same probability. Hence

$$\begin{aligned} \mathbb{E}(\Psi_i \mid X_{t_i}) &= \frac{(K - X_{t_i})X_{t_i}}{K^2} ((g(X_{t_i} - 1) - g(X_{t_i})) + (g(X_{t_i} + 1) - g(X_{t_i}))) \\ &= \frac{(K - X_{t_i})X_{t_i}}{K^2} ((g(X_{t_i} - 1) - g(X_{t_i})) - (g(X_{t_i}) - g(X_{t_i} + 1))) \\ &= \frac{(K - X_{t_i})X_{t_i}}{K^2} \cdot \sqrt{2K} \left( \frac{1}{\sqrt{X_{t_i}}} - \frac{1}{\sqrt{X_{t_i} + 1}} \right) \\ &\leq \frac{X_{t_i}}{K} \cdot \sqrt{2K} \left( \frac{1}{\sqrt{X_{t_i}}} - \frac{1}{\sqrt{X_{t_i} + 1}} \right). \end{aligned}$$

We estimate the bracketed terms using

$$\frac{1}{\sqrt{X_{t_i}}} - \frac{1}{\sqrt{X_{t_i} + 1}} = \frac{\sqrt{X_{t_i} + 1} - \sqrt{X_{t_i}}}{\sqrt{X_{t_i}}\sqrt{X_{t_i} + 1}} \leq \frac{1/(2\sqrt{X_{t_i}})}{X_{t_i}} \leq \frac{1}{(X_{t_i})^{3/2}},$$

where the last inequality exploited that  $f(x+h) - f(x) \leq hf'(x)$  for any concave, differentiable function  $f$  and  $h \geq 0$ ; here using  $f(x) = \sqrt{x}$  and  $h = 1$ . Altogether,

$$\mathbb{E}(\Psi_i \mid X_{t_i}) \leq \frac{X_{t_i}}{K} \cdot \frac{\sqrt{2K}}{(X_{t_i})^{3/2}} = \frac{\sqrt{2}X_{t_i}}{\sqrt{K}(X_{t_i})^{3/2}} \leq \sqrt{\frac{2}{X_{t_i}K}},$$

which proves (3) since  $X_{t_i} \geq 1$  and  $K = \omega(1)$ .

To verify the bound on the variance, note that

$$\begin{aligned}\text{Var}(\Psi_i \mid X_{t_i}) &\geq \mathbb{E}((\Psi_i - \mathbb{E}(\Psi_i \mid X_{t_i}))^2 \cdot \mathbf{1}\{\Psi_i \leq 0\} \mid X_{t_i}) \\ &\geq \mathbb{E}((\Psi_i)^2 \cdot \mathbf{1}\{\Psi_i \leq 0\} \mid X_{t_i})\end{aligned}$$

since  $\mathbb{E}(\Psi_i \mid X_{t_i}) \geq 0$ . Now, as  $0 < X_{t_i} \leq K/2$ , we have  $\mathbb{P}[Y_{t_i+1} < Y_{t_i}] = \frac{(K-X_{t_i})X_{t_i}}{K^2} \geq \frac{X_{t_i}}{2K}$ . Moreover,  $Y_{t_i+1} < Y_{t_i}$  implies that  $X_{t_i+1} = X_{t_i} + 1$  since  $g$  is monotone decreasing on  $\{1, \dots, K/2\}$  and the  $X_{t_i}$ -value can change by either  $-1$ ,  $0$ , or  $1$ . Hence, if  $Y_{t_i+1} < Y_{t_i}$  then  $Y_{t_i+1} - Y_{t_i} = g(X_{t_i} + 1) - g(X_{t_i}) = -\sqrt{2K/(X_{t_i} + 1)}$ . Altogether,

$$\text{Var}(\Psi_i \mid X_{t_i}) \geq \frac{X_{t_i}}{2K} \cdot \left(-\sqrt{2K/(X_{t_i} + 1)}\right)^2 \geq 1/4,$$

where we used  $X_{t_i}/(X_{t_i} + 1) \geq 1/2$ . This proves the lower bound on the variance.

We are almost ready to prove that  $Y_{t_i} := \sum_{j=0}^{i-1} \Psi_j$  can be approximated by a normally distributed random variable for sufficiently large  $t$ . We denote by  $s_i^2 := \sum_{j=0}^{i-1} \text{Var}(\Psi_j \mid X_{t_j})$  and note that  $s_i^2 \geq i/4$  by our analysis of variance from above. The so-called Lyapunov condition, which is sufficient for convergence to the normal distribution (see Lemma 8), requires the existence of some  $\delta > 0$  such that

$$\lim_{i \rightarrow \infty} \frac{1}{s_i^{2+\delta}} \sum_{j=0}^{i-1} \mathbb{E}(|\Psi_j - \mathbb{E}(\Psi_j \mid X_{t_j})|^{2+\delta} \mid X_{t_j}) = 0.$$

We will show that the condition is satisfied for  $\delta = 1$  (smaller values could be used but do not give any benefit) and  $i = \omega(K)$  (which, as  $i = \alpha s^2 K^2$ , holds due to our assumptions  $1/\alpha = o(K)$  and  $|s| = \Omega(1)$ ). We argue that

$$|\Psi_i - \mathbb{E}(\Psi_i \mid X_{t_i})| \leq |\Psi_i| + |\mathbb{E}(\Psi_i \mid X_{t_i})| \leq |\max\{k \mid \mathbb{P}[|\Psi_i| \geq k \mid X_{t_i}] > 0\}| + o(1),$$

where we have used the bound on  $|\mathbb{E}(\Psi_i \mid X_{t_i})|$  from (3). As the  $X_{t_i}$ -value can only change by  $\{-1, 0, 1\}$ , we get, by summing up all possible changes of the  $g$ -value, that

$$\begin{aligned}|\Psi_i - \mathbb{E}(\Psi_i \mid X_{t_i})| &\leq (g(X_{t_i} - 1) - g(X_{t_i})) + (g(X_{t_i}) - g(X_{t_i} + 1)) + o(1) \\ &\leq g(X_{t_i} - 1) - g(X_{t_i} + 1) + o(1) \\ &\leq \left(2 \cdot \sqrt{2K/(X_{t_i} - 1)}\right) + o(1)\end{aligned}$$

for  $K$  large enough.

Hence, plugging this in the Lyapunov condition,

$$\mathbb{E}(|\Psi_j - \mathbb{E}(\Psi_j \mid X_{t_j})|^3 \mid X_{t_j}) \leq \frac{2X_{t_j}}{K} \left(2 \cdot \sqrt{2K/(X_{t_j} - 1)}\right)^3 (1 + o(1)) + o(1) = O(\sqrt{K}),$$

implying that

$$\frac{1}{s_i^3} \sum_{j=0}^{i-1} \mathbb{E}(|\Psi_j - \mathbb{E}(\Psi_j)|^3 \mid X_{t_j}) \leq \frac{1}{(i/4)^{1.5}} O(i\sqrt{K}) = O(\sqrt{K/i}),$$



which goes to 0 as  $i = \omega(\sqrt{K})$ . Hence, for the value  $i := \alpha s^2 K^2$  considered in the lemma we obtain that  $\frac{Y_{t_i} - \mathbb{E}(Y_{t_i} | X_0)}{s_i}$  converges in distribution to  $N(0, 1)$ . Note that  $s_i^2 \geq \alpha s^2 K^2 / 4$  by our analysis of variance and therefore  $s_i \geq \sqrt{\alpha} |s| K / 2$ . We have to be careful when computing  $\mathbb{E}(Y_{t_i})$  since  $\mathbb{E}(\Psi_i | X_{t_i})$  is negative for  $X_{t_i} > K/2$ . Note, however, that considerations are stopped when the marginal probability exceeds  $5/6$ , i. e., when  $X_{t_i} > 5K/6$ . Using (3), we hence have that  $\mathbb{E}(\Psi_i | X_{t_i}) \geq -\sqrt{2/(5K^2/6)} \geq -1.55/K$ . Therefore,  $\mathbb{E}(Y_{t_i}) \geq i \cdot (-1.55/K) = -1.55\alpha s^2 K$  and  $\mathbb{E}(Y_{t_i}/s_i) \geq -3.1|s|\sqrt{\alpha}$ .

Hence, using the approximation by the normal distribution and taking into account the scaling and shifting, we have that

$$\begin{aligned} \mathbb{P}[Y_{t_i} \geq rK] \\ \geq (1 - o(1))(1 - \Phi(rK/s_i - \mathbb{E}(Y_{t_i}/s_i))) = (1 - o(1))(1 - \Phi(r/(|s|\sqrt{\alpha/4}) + 3.1|s|\sqrt{\alpha})) \end{aligned} \quad (5)$$

for any  $r$  leading to a positive argument of  $\Phi$ , where  $\Phi$  denotes the cumulative distribution function of the standard normal distribution.

Recall that our aim is to bound  $\sum_{j=0}^{i-1} \Delta_j = X_{t_i} - X_0$ . To this end, we look into the event that  $Y_{t_i} \geq 3\sqrt{|s|}K$  (noting that  $s < 0$ ) and study  $Y_{t_i} - g(X_{t_i}) < 0$ , which reflects the accumulated effect of b-steps on the potential function until time  $t_i$  (recall that a b-step increases the  $X_t$ -value and decreases the  $g(X_t)$ -value. Given  $X_t = x$  and assuming a b-step at time  $t$ , we have  $X_{t+1} > x$  with probability at most  $x/K$ . Hence,  $g(X_{t+1}) - g(x) \geq -\frac{x}{K} \frac{2\sqrt{K}}{\sqrt{x}} \geq -2$  since  $x \leq K/2$ . Let  $B := \{j \leq t_i \mid j \notin \{t_1, \dots, t_i\}\}$  be the indices of the b-steps until time  $t_i$ . Since by assumption only  $|s|K/4$  b-steps occur until time  $t_i$ , we get  $\sum_{j \in B} \mathbb{E}(g(X_{j+1}) - g(X_j) \mid X_j) \geq -|s|K/2$ , and, with probability at least  $1/2$ ,  $\sum_{j \in B} g(X_{j+1}) - g(X_j) \geq -|s|K$  using Markov's inequality (noting that all terms are non-positive, so Markov's inequality can be applied to the negative of the random variable). We assume this to happen, which accounts for the factor  $1/2$  in the statement of the lemma. Thus, by combining the effect of the rw-steps with the b-steps we obtain  $g(X_{t_i}) - g(X_0) \geq 3\sqrt{|s|}K - |s|K \geq 3\sqrt{|s|}K - \sqrt{|s|}K \geq 2\sqrt{|s|}K$ .

Finally, from (2), we know that  $g(X_{t_i}) - g(X_0) \geq 2\sqrt{|s|}K$  implies that  $X_{t_i} - X_0 \leq sK < 0$ , hence clearly  $\sum_{j=0}^i (X_{t_{j+1}} - X_{t_j}) \leq sK$ . By (5) and Lemma 18 (in the appendix), to bound  $\mathbb{P}[Y_{t_i} \geq 3\sqrt{|s|}K]$  from below, we compute

$$\frac{3\sqrt{|s|}}{|s|\sqrt{\alpha/4}} + 3.1|s|\sqrt{\alpha} \leq \frac{13}{\sqrt{|s|\alpha}}$$

using  $|s| \leq 1$  and  $\alpha \leq 1$ , and get

$$\left( \frac{1}{13\sqrt{1/(|s|\alpha)}} - \frac{1}{(13\sqrt{1/(|s|\alpha)})^3} \right) \frac{1}{\sqrt{2\pi}} e^{-169/(2|s|\alpha)} =: p(\alpha, s).$$

This means that distance  $sK$  (in negative direction) is bridged by the rw-steps before or at time  $t_i$ , where  $i = \alpha s^2 K^2$ , with probability at least  $(1/2 - o(1))p(\alpha, s)$ , where the

factor  $1/2$  comes from the application of Markov's inequality. Undoing the scaling of the state space, this corresponds to an accumulated change of the actual state of cGA in rw-steps by  $s$ ; more formally,  $(\sum_{i=0}^t \Delta_i) \leq s$  in terms of the original state space. This establishes also the first statement of the lemma and completes the proof.  $\square$

Lemma 7 requires a bounded number of b-steps. To establish this, we first show that, during the early stages of a run, the probability of a b-step is only  $O(1/\sqrt{n})$ . Intuitively, during early stages of the run many bits will have marginal probabilities in the interval  $[1/6, 5/6]$ . Then the standard sampling deviation of the ONEMAX-value is of order  $\Theta(\sqrt{n})$ , and the probability of a b-step is  $1 - P[R_t] = O(1/\sqrt{n})$ . The link between  $1 - P[R_t]$  and the standard deviation already appeared in Lemma 3 above; roughly, it says that every step is a b-step for bit  $i$  with probability at least  $(\sum_{j \neq i} p_j(1 - p_j))^{-1/2}$ , which is the reciprocal of the standard deviation in terms of the other bits.

The following lemmas, most notably Lemma 9 and Lemma 11, represent a kind of counterpart of Lemma 3, but here we seek an *upper* bound on  $1 - P[R_t]$ . The analysis in Lemma 9 is non-trivial and uses advanced lemmas on properties of the binomial distribution, including Schur-convexity. Lemma 11 then applies the general Lemma 9 to bound the probability of a b-step.

**Lemma 9.** *Let  $S$  be the sum of  $m$  independent Poisson trials with probabilities  $p_1, \dots, p_m$  such that  $1/6 \leq p_i \leq 5/6$  for all  $1 \leq i \leq m$ . Then we have that for all  $0 \leq s \leq m$ ,*

$$\Pr(S = s) = O(1/\sqrt{m}).$$

**Proof.** Samuels [21] showed that  $\Pr(S = s)$  is nondecreasing in  $s \leq E(S)$  and nonincreasing in  $s \geq E(S)$ , hence it is maximal for  $s \in \{\lfloor E(S) \rfloor, \lceil E(S) \rceil\}$ . Hence for every  $0 \leq s \leq m$ ,

$$\begin{aligned} \Pr(S = s) &\leq \max\{\Pr(S = \lfloor E(S) \rfloor), \Pr(S = \lceil E(S) \rceil)\} \\ &\leq \Pr(\lfloor E(S) \rfloor - 2 \leq S \leq \lceil E(S) \rceil + 1). \end{aligned}$$

As remarked in [15, page 496], the above is Schur-convex in  $p_1, \dots, p_m$ ; this statement goes back to Gleser [11]. Hence the above probability is maximized if the vector of probabilities  $(p_1, \dots, p_m)$  is a maximal element w.r.t. the preorder of majorization, i.e. for a fixed sum of  $E(S) = \sum_{i=1}^m p_i$ , all probabilities  $p_i$  are at their respective borders:  $p_i \in \{1/6, 5/6\}$ , except for potentially one probability.

Let  $p'_1, \dots, p'_m$  denote such a best case distribution, i.e. for  $S' := \sum_{i=1}^m p'_i$  we have  $E(S') = E(S)$  and  $p'_1 = \dots = p'_k = 1/6$ ,  $p'_{k+1} = \dots = p'_{m-1} = 5/6$  and  $p'_m \in [1/6, 5/6]$ . Then

$$\begin{aligned} \Pr(S = s) &\leq \Pr(\lfloor E(S) \rfloor - 2 \leq S \leq \lceil E(S) \rceil + 1) \\ &\leq \Pr(\lfloor E(S') \rfloor - 2 \leq S' \leq \lceil E(S') \rceil + 1) \end{aligned}$$

Now assume that  $k \geq (m-1)/2$ . We apply the principle of deferred decisions and assume that the values of bits  $k+1, \dots, m$  are known. Let  $Y_k := \sum_{i=1}^k y_i$  be the number

of ones on the  $k$  bits having a marginal probability of  $1/6$ . Note that there is at most one value of  $Y_k$  that leads to a particular value of  $S'$ . Bounding all such probabilities by the probability of the mode,  $\max_y \Pr(Y_k = y)$ , we get

$$\Pr(\lfloor E(S') \rfloor - 2 \leq S' \leq \lceil E(S') \rceil + 1) \leq 5 \max_y \Pr(Y_k = y).$$

Since  $Y_k$  follows a Binomial distribution with parameters  $k$  and  $1/6$ , its mode is either  $\lfloor k/6 \rfloor$  or  $\lceil k/6 \rceil$ . Using bounds on binomial coefficients (Corollary 2.3 in [22] for  $m = 1$ ), it is easy to show (see Lemma 10 below) that  $\Pr(Y_k = \lfloor k/6 \rfloor)$  and  $\Pr(Y_k = \lceil k/6 \rceil)$  are both bounded by  $O(1/\sqrt{k})$ , hence  $\max_y \Pr(Y_k = y) = O(1/\sqrt{k}) = O(1/\sqrt{n})$ , and

$$\Pr(S = s) \leq 5 \max_y \Pr(Y_k = y) = O(1/\sqrt{n}).$$

The case  $k < (m - 1)/2$  is symmetric; we then consider the Binomial distribution over bits  $k + 1, \dots, m - 1$ , of which there are at least  $m - 1 - k \geq (m - 1)/2$  many.  $\square$

The following lemma is used to bound the mode of the binomial distribution in the proof of Lemma 9 above.

**Lemma 10.** *Let  $X \sim \text{Bin}(n, p)$  for some  $0 < p < 1$ . If  $np$  is an integer then*

$$\Pr[X = np] \leq \frac{1}{\sqrt{2\pi np(1-p)}}.$$

*Otherwise,*

$$\Pr[X = \lceil np \rceil] \leq \frac{e}{\sqrt{2\pi np(1+a)(1-p(1+a))}}$$

*for  $a = \frac{\lceil np \rceil}{np} - 1 \leq 1/(np)$ , and*

$$\Pr[X = \lfloor np \rfloor] \leq \frac{e}{\sqrt{2\pi np(1-a)(1-p(1-a))}}$$

*for  $a = 1 - \frac{\lfloor np \rfloor}{np} \leq 1/(np)$ .*

**Proof.** We start with the integral case. By definition,

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

We use the following bound on the binomial coefficient (Corollary 2.3 in [22] for  $m = 1$ ):

$$\binom{n}{\alpha n} < \frac{1}{\sqrt{2\pi n \alpha(1-\alpha)} \alpha^{\alpha n} (1-\alpha)^{n-\alpha n}}.$$

Plugging this in the formula for  $\Pr[X = k]$  with  $k = np$  and  $\alpha = p$ , we get the desired result if  $np$  is integer.

If  $np$  is not an integer, then we write  $\lceil np \rceil = np(1+a)$  for some  $a \leq 1/(np)$ . Following the same approach with  $k = np(1+a)$  and  $\alpha = p(1+a)$ , we get

$$\begin{aligned}
& \mathbb{P}[X = \lceil np \rceil] \\
& \leq \frac{p^{np(1+a)}(1-p)^{n-np(1+a)}}{\sqrt{2\pi np(1+a)(1-p(1+a))}((1+a)p)^{np(1+a)}(1-p(1+a))^{n-np(1+a)}} \\
& \leq \frac{1}{\sqrt{2\pi np(1+a)(1-p(1+a))}} \left(1 + \frac{pa}{1-p(1+a)}\right)^{n-np(1+a)} \\
& \leq \frac{1}{\sqrt{2\pi np(1+a)(1-p(1+a))}} e^{\frac{pa}{1-p(1+a)}(n-np(1+a))} \leq \frac{e}{\sqrt{2\pi np(1+a)(1-p(1+a))}},
\end{aligned}$$

where the second inequality bounded  $(p/(p(1+a)))^{np(1+a)} \leq 1$ , the third used  $1+x \leq e^x$  and the fourth  $pan \leq 1$ . The bound for  $\mathbb{P}[X = \lfloor np \rfloor]$  is proved analogously, with  $1-a$  taking the role of  $1+a$  and the roles of  $p$  and  $1-p$  swapped.  $\square$

We remark that if  $p = 1/2$  and  $np$  integer, we recover from the previous lemma the following well-known bound on the central binomial coefficient:  $\binom{n}{n/2} \leq 2^{n+1/2}/(\sqrt{\pi n})$ .

**Lemma 11.** *Assume that at time  $t$  there are  $\gamma n$  bits for some constant  $\gamma > 0$  bits whose marginal probabilities are within  $[1/6, 5/6]$ . Then the probability of having a b-step on any fixed bit position is*

$$1 - \mathbb{P}[R_t] = O(1/\sqrt{n}),$$

*regardless of the decisions made in this step on all other  $n - \gamma n - 1$  bits.*

**Proof.** We know from our earlier discussion that a b-step at bit  $i$  requires  $D_t \in \{-1, 0\}$  where  $D_t := |x| - |x_i| - (|y| - |y_i|)$  is the change of the ONEMAX-value at bits other than  $i$  in the two solutions  $x$  and  $y$  sampled at time  $t$ .

We apply the principle of deferred decisions and fix all decisions for creating  $x$  as well as decisions for  $y$  on all but the  $m := \gamma n$  selected bits with marginal probabilities in  $[1/6, 5/6]$ . Let  $p_1, p_2, \dots, p_m$  denote the corresponding marginal probabilities after renumbering these bits, and let  $S$  denote the random number of these bits set to 1. Note that there are at most 2 values for  $S$  which lead to the algorithm making a b-step.

Since  $S$  is determined by a Poisson trial with success probabilities  $p_1, \dots, p_m$ , Lemma 9 implies that the probability of  $S$  attaining any particular value is  $O(1/\sqrt{m}) = O(1/\sqrt{n})$ . Taking the union bound over 2 values proves the claim.  $\square$

Even though one main aim is to show that rw-steps make certain marginal probabilities reach their lower border, we will also ensure that with high probability,  $\Omega(n)$  marginal probabilities do not move by too much, resulting in a large sampling variance and a small probability of b-steps. The following lemma serves this purpose. Its proof is a straightforward application of Hoeffding's inequality since it is pessimistic here to ignore the self-loops.

**Lemma 12.** *For any bit, with probability  $\Omega(1)$  for any  $t \leq \kappa K^2$ ,  $\kappa > 0$  a small enough constant, the first  $t$  rw-steps lead to a total change of the bit's marginal probability within  $[-1/6, 1/6]$ . This fact holds independently of all other bits.*

*The probability that the above holds for less than  $\gamma n$  bits amongst the first  $n/2$  bits is  $2^{-\Omega(n)}$ , regardless of the decisions made on the last  $n/2$  bits.*

**Proof.** Note that the probability of exceeding  $[-1/6, 1/6]$  increases with the number of rw-steps that do increase or decrease the marginal probability (as opposed to self-loops). We call these steps *relevant* and pessimistically assume that all  $t$  steps are relevant steps.

Now defining  $X_j := \sum_{i=1}^j X_i$  as the total progress in the first  $j$  relevant steps, we have  $E(X_j) = 0$ , for all  $j \leq t$ , and the total change in these  $j$  steps exceeds  $1/6$  only if  $X_j \geq K/6$ . Applying a Hoeffding bound, Theorem 1.13 in [4], the maximum total progress is bounded as follows:

$$\Pr\left(\max_{j \leq t} X_j \leq K/6\right) \leq \exp\left(\frac{-2(K/6)^2}{4t}\right) \leq \exp\left(-\frac{1}{12\kappa}\right).$$

By symmetry, the same holds for the total change reaching values less or equal to  $-1/6$ . By the union bound, the probability that the total change always remains within the interval  $[-1/6, 1/6]$  is thus at least

$$1 - 2 \exp\left(-\frac{1}{12\kappa}\right).$$

Assuming  $\kappa < 1/(12 \ln 2)$  gives a lower bound of  $\Omega(1)$ .

Note that due to our pessimistic assumption of all steps being relevant, all bits are treated independently. Hence we may apply standard Chernoff bounds to derive the second claim.  $\square$

The following lemma shows that whenever a small number of bits has reached the lower border for marginal probabilities, the remaining optimization time is  $\Omega(n \log n)$  with high probability. The proof is similar to the well known coupon collector's theorem [16].

**Lemma 13.** *Assume cGA reaches a situation where at least  $\Omega(n^\varepsilon)$  marginal probabilities attain the lower border  $1/n$ . Then with probability  $1 - e^{-\Omega(n^{\varepsilon/2})}$ , and in expectation, the remaining optimization time is  $\Omega(n \log n)$ .*

**Proof.** Let  $m$  be the number of bits that have reached the lower border  $1/n$ . A necessary condition for reaching the optimum within  $t := (n/2 - 1) \cdot (\varepsilon/2) \ln n$  iterations is that during this time each of these  $m$  bits is sampled at value 1 in at least one of the two search points constructed. The probability that one bit never samples a 1 in  $t$  iterations is at least  $(1 - 2/n)^t$ . The probability that all  $m$  bits sample a 1 during  $t$  steps is at most, using  $(1 - 2/n)^{n/2-1} \geq 1/e$  and  $1 + x \leq e^x$  for  $x \in \mathbb{R}$ ,

$$\left(1 - \left(1 - \frac{2}{n}\right)^t\right)^m \leq \left(1 - n^{-\varepsilon/2}\right)^m \leq \exp(-\Omega(n^{\varepsilon/2})).$$

Hence with probability  $1 - \exp(-\Omega(n^{\varepsilon/2}))$  the remaining optimization time is at least  $t = \Omega(n \log n)$ . As  $1 - \exp(-\Omega(n^{\varepsilon/2})) = \Omega(1)$ , the expected remaining optimization time is of the same order.  $\square$

We have collected most of the machinery to prove Theorem 5. The following lemma identifies a set of bits that stay centered in a phase of  $\Theta(K \min\{K, \sqrt{n}\})$  steps, resulting in a low probability of b-steps. Basically, the idea is to bound the accumulated effect of b-steps in the phase using Chernoff bounds: given  $K/6$  b-steps, a marginal probability cannot change by more than  $1/6$ . Note that this applies to many, but not all bits. Later, we will see that within the phase, some of the remaining bits will reach their lower border with not too low probability.

**Lemma 14.** *Let  $\kappa > 0$  be a small constant. There exists a constant  $\gamma$ , depending on  $\kappa$ , and a selection  $S$  of  $\gamma n$  bits among the first  $n/2$  bits such that the following properties hold regardless of the last  $n/2$  bits throughout the first  $T := \kappa K \cdot \min\{K, \sqrt{n}\}$  steps of  $cGA$  with  $K \leq \text{poly}(n)$ , with probability  $\text{poly}(n) \cdot 2^{-\Omega(\min\{K, n\})}$ :*

1. *the marginal probabilities of all bits in  $S$  is always within  $[1/6, 5/6]$  during the first  $T$  steps,*
2. *the probability of a b-step at any bit is always  $O(1/\sqrt{n})$  during the first  $T$  steps, and*
3. *the total number of b-steps for each bit is bounded by  $K/6$ , leading to a displacement of at most  $1/6$ .*

**Proof.** The first property is trivially true at initialization, and we show that an event of exponentially small probability needs to occur in order to violate the property. Taking a union bound over all  $T$  steps ensures that the property holds throughout the whole phase of  $T$  steps with the claimed probability.

By Lemma 12, with probability  $1 - 2^{-\Omega(n)}$ , for at least  $\gamma n$  of these bits the total effect of all rw-steps is always within  $[-1/6, +1/6]$  during the first  $T \leq \kappa K^2$  steps. We assume in the following that this happens and take  $S$  as a set containing exactly  $\gamma n$  of these bits.

It remains to show that for all bits in  $S$  the total effect of b-steps is bounded by  $1/6$  with high probability. Note that, while this is the case, according to Lemma 11, the probability of a b-step at every bit in  $S$  is at most  $c_2/\sqrt{n}$  for a positive constant  $c_2$ . This corresponds to the second property, and so long as this holds, the expected number of b-steps in  $T \leq \kappa K^2$  steps is at most  $\kappa \cdot c_2 K$ . Each b-step changes the marginal probability of the bit by  $1/K$ . A necessary condition for increasing the marginal probability by a total of at least  $1/6$  is that we have at least  $K/6$  b-steps amongst the first  $T$  steps. Choosing  $\kappa$  small enough to make  $\kappa \cdot c_2 K \leq 1/2 \cdot K/6$ , by Chernoff bounds the probability to get at least  $K/6$  b-steps in  $T$  steps is  $e^{-\Omega(K)}$ . In order for the first property to be violated, an event of probability  $e^{-\Omega(K)}$  is necessary for any bit in  $S$  and any length of time  $t \leq T$ ; otherwise all properties hold true.

Taking the union bound over all  $T \leq \kappa K^2$  steps and all  $\gamma n$  bits gives a probability bound of  $\kappa K^2 \cdot \gamma n \cdot e^{-\Omega(K)} \leq \text{poly}(n) \cdot 2^{-\Omega(K)}$  for a property being violated. This proves the claim.  $\square$

Finally, we put everything together to prove our lower bound for cGA.

**Proof of Theorem 5.** If  $K = O(1)$  then it is easy to show, similarly to Lemma 16, that each bit independently hits the lower border with probability  $\Omega(1)$  by sampling only zeros. Then the result follows easily from Chernoff bounds and Lemma 13. Hence we assume in the following  $K = \omega(1)$ .

For  $K \geq \sqrt{n}$ , Lemma 14 implies a lower bound of  $\Omega(K\sqrt{n})$  as then the probability of sampling the optimum in any of the first  $T := \kappa K \cdot \min\{K, \sqrt{n}\}$  steps is at most  $(5/6)^{\gamma n} = 2^{-\Omega(n)}$ . Taking a union bound over the first  $T$  steps and adding the error probability from Lemma 14 proves the claim for a lower bound of  $\Omega(K\sqrt{n})$  with the claimed probability. This proves the theorem for  $K = \Omega(\sqrt{n} \log n)$  as then the  $\Omega(\sqrt{n}K)$  term dominates the runtime. Hence we may assume  $K = o(\sqrt{n} \log n)$  in the following and note that in this realm proving a lower bound of  $\Omega(n \log n)$  is sufficient as here this term dominates the runtime.

We still assume that the events from Lemma 14 apply to the first  $n/2$  bits. We now use Lemma 7 to show that some marginal probabilities amongst the last  $n/2$  bits are likely to walk down to the lower border. Note that Lemma 7 applies for an arbitrary (even adversarial) mixture of rw-steps and b-steps over time, so long as the overall number of b-steps is bounded. This allows us to regard the progress in rw-steps as independent between bits.

In more detail, we will apply both statements of Lemma 7 to a fresh marginal probability from the last  $n/2$  bits, to prove that it walks to its lower border with a not too small probability. First we apply the second statement of the lemma for a positive displacement of  $s := 1/6$  within  $T$  steps, using  $\alpha := T/((sK)^2)$ . The random variable  $T_s$  describes the first point of time where the marginal probability reaches a value of at least  $1/2 + 1/6 + s = 5/6$  through a mixture of b- and rw-steps. This holds since we work under the assumption that the b-steps only account for a total displacement of at most  $1/6$  during the phase. Lemma 7 now gives us a probability of at least  $1 - e^{-1/(4\alpha)} = \Omega(1)$  (using  $\alpha = O(1)$ ) for the event that the marginal probability does not exceed  $5/6$ . In the following, we condition on this event.

We then revisit the same stochastic process and apply Lemma 7 again to show that, under this condition, the random walk achieves a negative displacement. Note that the event of not exceeding a certain positive displacement is positively correlated with the event of reaching a given negative displacement (formally, the state of the conditioned stochastic process is always stochastically smaller than of the unconditioned process), allowing us to apply Lemma 7 again despite dependencies between the two applications.

We can therefore apply the first statement of Lemma 7 for a negative displacement of  $s := -5/6$  within  $T$  steps, still using  $\alpha := T/((sK)^2)$ . Note that by Lemma 14 at most  $K/6 \leq |s|K/4$  steps are b-steps. The conditions on  $\alpha$  hold as  $0 < \alpha < 1$  choosing  $\kappa$  small enough, and  $1/\alpha = O(K/\min\{\sqrt{n}, K\}) = o(K)$  for  $K = \omega(1)$ . Also note that

$1/\alpha = O(K/\min\{\sqrt{n}, K\}) = o(\log n)$  since  $K = o(n \log n)$ . Now Lemma 7 states that the probability of the random walk reaching a total displacement of  $-5/6$  (or hitting the lower border before) is at least

$$\begin{aligned} & \left(\frac{1}{2} - o(1)\right) \left( \frac{1}{13\sqrt{1/(|s|\alpha)}} - \frac{1}{(13\sqrt{1/(|s|\alpha)})^3} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{169}{2|s|\alpha}} \\ &= \Omega\left(\frac{1}{o(\sqrt{\log n})} \cdot e^{-o(\ln n)}\right) \geq n^{-\beta} \end{aligned}$$

for some  $\beta = o(1)$ . Combining with the probability of not exceeding  $5/6$ , the probability of the bit's marginal probability hitting the lower border within  $T$  steps is  $\Omega(n^{-\beta})$ . Hence by Chernoff bounds, with probability  $1 - 2^{-\Omega(n^{1-\beta})}$ , the final number of bits hitting the lower border within  $T$  steps is  $\Omega(n^{1-\beta}) = \Omega(n^{1-o(1)})$ .

Once a bit has reached the lower border, while the probability of a b-step is  $O(1/\sqrt{n})$ , the probability of leaving the bound again is  $O(n^{-3/2})$  as it is necessary that either the bit is sampled as 1 at one of the offspring and a b-step happens, or in both offspring the bit is sampled as 1. So the probability that this does not happen until the  $T = O(n \log n)$  steps are completed is  $(1 - O(n^{-3/2}))^T \leq e^{-O(\log(n)/\sqrt{n})} = o(1)$ . Again applying Chernoff bounds leaves  $\Omega(n^{1-o(1)})$  bits at the lower border at time  $T$  with probability  $1 - 2^{-\Omega(n^{1-o(1)})}$ .

Then Lemma 13 implies a lower bound of  $\Omega(n \log n)$  that holds with probability  $1 - 2^{-\Omega(n^{1/2-o(1)})}$ .  $\square$

## 5.2 Proof of Lower Bound for 2-MMAS<sub>ib</sub>

We will use, to a vast extent, the same approach as in Section 5.1 to prove Theorem 6. Most of the lemmas can be applied directly or with very minor changes. In particular, Lemma 12, Lemma 13 and Lemma 14 also apply to 2-MMAS<sub>ib</sub> by identifying  $1/K$  with  $\rho$ . Intuitively, this holds since the analyses of b-steps always pessimistically bound the absolute change of a marginal probability by the update strength ( $1/K$  for cGA). This also holds with respect to the update strength  $\rho$  for 2-MMAS<sub>ib</sub>.

To prove lower bounds on the time to hit a border through rw-steps, the next lemma is used. It is very similar to Lemma 7, except for two minor differences: first, also the accumulated effect of b-steps is included in the quantity  $p_t - p_0$  analyzed in the lemma. Second, considerations are stopped when the marginal probability becomes less than  $\rho$  or more than  $1 - \rho$ . This has technical reasons but is not a crucial restriction. We supply an additional lemma, Lemma 16 below, that applies when the marginal probability is less than  $\rho$ . The latter lemma uses known analyses similar to so-called landslide sequences defined in [18, Section 4].

**Lemma 15.** *Consider a bit of 2-MMAS<sub>ib</sub> on ONEMAX and let  $p_t$  be its marginal probability at time  $t$ . We say that the process breaks a border at time  $t$  if  $\min\{p_t, 1 - p_t\} \leq \max\{1/n, \rho\}$ . Given  $s \in \mathbb{R}$  and arbitrary starting state  $p_0$ , let  $T_s$  be the smallest  $t$  such that  $\text{sgn}(s)(p_t - p_0) \geq |s|$  holds or a border is broken.*



Choosing  $0 < \alpha < 1$ , where  $1/\alpha = o(\rho^{-1})$ , and  $-1 < s < 0$  constant, and assuming that every step is a b-step with probability at most  $\rho/(4\alpha)$ , we have

$$\begin{aligned} & \mathbb{P}[T_s \leq \alpha(s/\rho)^2 \text{ or } p_t \text{ exceeds } 5/6 \text{ before } T_s] \\ & \geq (1 - o(1)) \cdot \left( \frac{1}{\sqrt{(24/(|s|\alpha))}} - \frac{1}{(24/(|s|\alpha))^3} \right) \frac{1}{\sqrt{2\pi}} e^{-288/(|s|\alpha)}. \end{aligned}$$

Moreover, for any  $\alpha > 0$  and constant  $0 < s < 1$ , if there are at most  $s/(2\alpha\rho)$  b-steps until time  $\alpha(s/\rho)^2$ , then

$$\mathbb{P}[T_s \geq \alpha(s/\rho)^2 \text{ or a border is broken until time } \alpha(s/\rho)^2] \geq 1 - e^{-1/(16\alpha)}.$$

**Proof.** We follow similar ideas as in the proof of Lemma 7. Again, we start with the second statement, where  $s \geq 0$  is assumed, and aim for applying a Hoeffding bound. We note that a marginal probability of 2-MMAS<sub>ib</sub> can only change by an absolute amount of at most  $\rho$  in a step. Hence, the b-steps until time  $\alpha(s/\rho^2)$  account for an increase of the  $X_t$ -value by at most  $s/2$ . With respect to the rw-steps, Theorem 1.11 from [4] can be applied with  $c_i = 2\rho$  and  $\lambda = s/2$ .

Also for the first statement, we follow the ideas from the proof of Lemma 7. In particular, the borders stated in the lemma will be ignored as all considerations are stopped when they are reached. We will apply a potential function and estimate its first and second moment separately with respect to rw-steps and non-rw steps.

Our potential function is

$$g(x) := \int_x^{1/2} \frac{1}{\rho\sqrt{z}} dz,$$

which can be considered the continuous analogue of the function  $g$  used in the proof of Lemma 7. For  $r > 0$  and  $x \leq 1/2$ , we note that

$$g(x - r) - g(x) = \frac{2}{\rho}(\sqrt{x} - \sqrt{x - r}). \quad (6)$$

For better readability, we denote by  $X_t := p_t$ ,  $t \geq 0$ , the stochastic process obtained by listing the marginal probabilities of the considered bit over time. Let  $Y_t := g(X_t)$  and  $\Delta_t := Y_{t+1} - Y_t$ . In the remainder of this proof, we assume  $X_t \leq 1/2$ ; analyses for the case  $X_t > 1/2$  are symmetrical by switching the sign of  $\Delta_t$ . We also assume  $X_t \geq \rho$  as we are only interested in statements before the first point of time where a border is broken.

We claim for all  $t \geq 0$  where rw-steps occur (hence, formally we enter the conditional probability space on  $R_t$ , the event that an rw-step occurs at time  $t$ ) that

$$0 \leq \mathbb{E}(\Delta_t \mid X_t; R_t) \leq \frac{3\rho}{2\sqrt{X_t}} = o(1) \quad (7)$$

$$\text{Var}(\Delta_t \mid X_t; R_t) \geq 1/16. \quad (8)$$

We start with the bounds on the expected value. Note that by the properties of rw-steps for 2-MMAS<sub>ib</sub>, where there are two possible successor states, we get the martingale property  $E(X_{t+1} | X_t) = (1 - X_t)(X_t - \rho X_t) + X_t(X_t + \rho(1 - X_t)) = X_t$ . Since  $g(x)$  is a convex function on  $[0, 1/2]$ , we have by Jensen's inequality  $E(\Delta_t | X_t) = E(g(X_{t+1}) | X_t) - g(X_t) \geq g(E(X_{t+1} | X_t)) - g(X_t) = 0$ . To bound the expected value from above, we carefully estimate the error introduced by the convexity. Note that

$$g(x - x\rho) - g(x) = \int_{x-x\rho}^x \frac{1}{\rho\sqrt{z}} dz \leq \frac{x}{\sqrt{x-x\rho}} \quad (9)$$

since the integrand is non-increasing. Analogously,

$$\frac{1-x}{\sqrt{x+(1-x)\rho}} \leq g(x) - g(x+(1-x)\rho) \leq \frac{1-x}{\sqrt{x}} \quad (10)$$

Inspecting the  $g$ -values of two possible successor states of  $x := X_t$ , we get that

$$E(\Delta_t | X_t = x) = E(g(X_{t+1}) - g(x) | X_t = x) \quad (11)$$

$$\begin{aligned} &\leq (1-x) \frac{x}{\sqrt{x-x\rho}} - x \frac{1-x}{\sqrt{x+(1-x)\rho}} = (1-x)x \left( \frac{1}{\sqrt{x-x\rho}} - \frac{1}{\sqrt{x+(1-x)\rho}} \right) \\ &= (1-x)x \cdot \frac{\sqrt{x+(1-x)\rho} - \sqrt{x-x\rho}}{\sqrt{x+(1-x)\rho} \cdot \sqrt{x-x\rho}} \leq \frac{(1-x)x \frac{\rho}{2\sqrt{x-x\rho}}}{x-x\rho} \leq \frac{x\rho}{2(x/2)^{3/2}} \\ &\leq \frac{3\rho}{2\sqrt{x}}, \end{aligned} \quad (12)$$

where the third-last inequality estimated  $1-x \leq 1$  and used that  $f(z+\rho) - f(z) \leq \rho f'(z)$  for any concave, differentiable function  $f$  and  $\rho \geq 0$ ; here using  $f(z) = \sqrt{z}$  and  $z = x - \rho$ . The penultimate used  $\rho \leq 1/2$ . Since the final bound is  $O(\rho/\sqrt{x}) = o(1)$  due to our assumption on  $X_t \geq \rho$ , we have proved (7).

We proceed with the bound on the variance. Note that

$$\begin{aligned} \text{Var}(\Delta_t | X_t) &\geq E((\Delta_t - E(\Delta_t | X_t = x))^2 \cdot \mathbf{1}\{\Delta_t \leq 0\} | X_t = x) \\ &\geq E((\Delta_t)^2 \cdot \mathbf{1}\{\Delta_t \leq 0\} | X_t = x) \end{aligned}$$

since  $E(\Delta_t | X_t) \geq 0$ . We note that for  $X_t = x$ , we have  $P[X_{t+1} \geq x] = x$ . On  $X_{t+1} \geq x$ , we have  $\Delta_t < 0$ , which means  $P[\Delta_t < 0] = x$ . Now,  $|\Delta_t| = g(x + (1-x)\rho) - g(x) \geq \frac{1-x}{\sqrt{x+\rho(1-x)}} \geq \frac{1-x}{\sqrt{x+x(1-x)}} \geq \frac{1}{4\sqrt{x}}$ , where the penultimate inequality used  $\rho \leq x$  and the last one  $x \leq 1/2$ . Plugging this in, we get

$$\text{Var}(\Delta_t | X_t = x) \geq x \cdot \left( \frac{1}{4\sqrt{x}} \right)^2 \geq \frac{1}{16},$$

which completes the proof of (8) with respect to rw-steps.

We now consider the case that a b-step occurs at time  $t$ . We are only interested in bounding  $E(\Delta_t | X_t)$  from below now. Given  $X_t = x$ , we have  $X_{t+1} > x$  (which means

$\Delta_t < 0$ ) with probability at most  $1 - (1 - x)^2 = 1 - (1 - 2x + x^2) \leq 2x$ . With the remaining probability,  $X_{t+1} < x$ . Since  $X_{t+1} \leq x + \rho$ , we get

$$\mathbb{E}(\Delta_t \mid X_t = x; \overline{R_t}) \geq -2x \int_x^{x+\rho} \frac{1}{\rho\sqrt{z}} dz \geq -2\sqrt{x}. \quad (13)$$

Now, since by assumption a b-step occurs with probability at most  $\rho/(4\alpha)$ , the unconditional expected value of  $\Delta_t$  can be computed using the superposition equality. Combining (7) and (13), we get

$$\mathbb{E}(\Delta_t \mid X_t = x) \geq 0 - \frac{\rho}{4\alpha} 2\sqrt{x} \geq -\frac{\rho}{2\alpha}. \quad (14)$$

since  $x \leq 1$ . By the law of total probability, we get for the unconditional variance that

$$\text{Var}(\Delta_t \mid X_t) = \text{Var}(\Delta_t \mid X_t; R_t)P[R_t] + \text{Var}(\Delta_t \mid X_t; \overline{R_t})(1 - P[R_t]),$$

Since  $P[R_t] \geq 1/2$ , we altogether have for the unconditional variance that

$$\text{Var}(\Delta_t \mid X_t = x) \geq 1/32.$$

To apply the central limit theorem (Lemma 8) on the sum of the  $\Delta_t$ , we will verify the Lyapunov condition for  $\delta = 1$  (smaller values could be used but do not give any benefit) and  $t = \omega(1/\rho)$  (which, as  $t = \alpha(s/\rho)^2$ , holds due to our assumptions  $1/\alpha = o(\rho^{-1})$  and  $|s| = \Omega(1)$ ). We compute

$$\begin{aligned} & \mathbb{E}(|\Delta_t - \mathbb{E}(\Delta_t \mid X_t)|^3 \mid X_t) \\ & \leq P[\Delta_t > 0] \cdot (\Delta_t - \mathbb{E}(\Delta_t \mid X_t))^3 + P[\Delta_t < 0] \cdot (|\Delta_t| + |\mathbb{E}(\Delta_t \mid X_t)|)^3 \\ & \leq (1 - x) \left( \frac{x}{\sqrt{x - x\rho}} \right)^3 + x \cdot \left( \frac{1 - x}{\sqrt{x}} + \frac{3\rho}{2\sqrt{x}} + \frac{\rho}{2\alpha} \right)^3, \end{aligned}$$

where we again have used (9) and the upper bound from (10) with respect to the two outcomes of  $X_{t+1}$ . Moreover, we have used the bound  $\mathbb{E}(\Delta_t \mid X_t) \geq 0$  in the first term and  $\mathbb{E}(|\Delta_t| \mid X_t) \leq 3\rho/(2\sqrt{x}) + \rho/(2\alpha)$  in the second term, which is a crude combination of (12) and (14). As  $\rho \leq 1/2$  and  $\rho \leq x$  as well as  $\alpha \geq \rho$ , the expected value satisfies

$$\begin{aligned} \mathbb{E}(|\Delta_t - \mathbb{E}(\Delta_t \mid X_t)|^3 \mid X_t) & \leq \left( \frac{x}{\sqrt{x/2}} \right)^3 + x \left( O\left( \frac{1}{\sqrt{x}} + 3\sqrt{x} + \frac{1}{2} \right) \right)^3 \\ & \leq 1 + x \left( O\left( \frac{1}{\sqrt{x}} \right) \right)^3 = O(1/\sqrt{x}) = O(1/\sqrt{\rho}), \end{aligned}$$

where we used  $x \leq 1$  and  $x \geq \rho$ . Using  $s_t^2 := \sum_{j=0}^{t-1} \text{Var}(\Delta_j \mid X_j)$  in the notation of Lemma 8 and using that  $\text{Var}(\Delta_j \mid X_j) \geq 1/32$ , we get

$$\frac{1}{s_t^3} \sum_{j=0}^{t-1} \mathbb{E}(|\Psi_j - \mathbb{E}(\Psi_j)|^3 \mid X_j) \leq \frac{182}{t^{1.5}} O(t/\sqrt{\rho}) = O(\sqrt{1/(t\rho)}),$$

which goes to 0 as  $t = \omega(1/\rho)$ . This establishes the Lyapunov condition. Hence, for the value  $t := \alpha(s/\rho)^2$  considered in the lemma, we obtain that  $\frac{Y_t - \mathbb{E}(Y_t | X_0)}{s_t}$  converges in distribution to the normal distribution  $N(0, 1)$ . Note that  $s_t^2 \geq \alpha(s/\rho)^2/32$  since  $\text{Var}(\Delta_t | X_t) \geq 1/32$ . Hence,  $s_t = \sqrt{\alpha/32}(|s|/\rho)$ , recalling that  $s < 0$ . Moreover, as  $x \leq 5/6$  is assumed in this part of the lemma, by combining (12) and (14), we get  $\mathbb{E}(\Delta_t | X_t) \geq -\rho/(2\alpha) - \rho \cdot (3/2)\sqrt{6/5} \geq -\rho/(2\alpha) - 1.7\rho \geq -2.2\rho/\alpha$  and  $\mathbb{E}(Y_t) \geq t(-2.2\rho/\alpha) \geq -2.2s^2/\rho$ . Together, this means  $\frac{\mathbb{E}(Y_t)}{s_t} \geq -\frac{2.2s^2/\rho}{\sqrt{\alpha/32}(|s|/\rho)} \geq -\sqrt{155/\alpha}|s| \geq -\sqrt{155/\alpha}$  since  $|s| \leq 1$  and  $\alpha \leq 1$ . By the normalization to  $N(0, 1)$ , we have that

$$\mathbb{P}[Y_t \geq r] = \mathbb{P}\left[\frac{Y_t}{s_t} - \frac{\mathbb{E}(Y_t | X_0)}{s_t} \geq \frac{r}{s_t} - \frac{\mathbb{E}(Y_t | X_0)}{s_t}\right],$$

hence  $\mathbb{P}[Y_t \geq r] \geq (1 - o(1))(1 - \Phi(r\rho/(|s|\sqrt{\alpha/32}) + \sqrt{155/\alpha}))$  for any  $r$  leading to a positive argument of  $\Phi$ , where  $\Phi$  denotes the cumulative distribution function of the standard normal distribution. We are interested in the event that  $Y_t \geq 2\sqrt{|s|}/\rho$ , recalling that  $s < 0$  and  $X_{t+1} \geq X_t \iff Y_{t+1} \leq Y_t$ . We made this choice because the event  $Y_t = g(X_t) - g(X_0) \geq 2\sqrt{|s|}/\rho$  implies that  $X_t - X_0 \leq s$  by (6).

To compute the probability of the event  $Y_t \geq 2\sqrt{|s|}/\rho$ , we choose  $r = 2\sqrt{|s|}/\rho$  and get  $r\rho/(|s|\sqrt{\alpha/32}) + \sqrt{155/\alpha} \leq 24/\sqrt{|s|\alpha}$ . We get

$$\mathbb{P}[Y_t \geq 2\sqrt{|s|}/\rho] \geq (1 - o(1))(1 - \Phi(24/\sqrt{|s|\alpha})).$$

By Lemma 18,

$$1 - \Phi(24/\sqrt{|s|\alpha}) \geq \left(\frac{1}{24/\sqrt{|s|\alpha}} - \frac{1}{(24/\sqrt{|s|\alpha})^3}\right) \frac{1}{\sqrt{2\pi}} e^{-288/(|s|\alpha)} =: p(\alpha, s),$$

which means that distance  $s$  is bridged (in negative direction) before or at time  $\alpha(s/\rho)^2$  with probability at least  $(1 - o(1))p(\alpha, s)$ .  $\square$

The following lemma shows that a marginal probability of less than  $\rho$  is unlikely to be increased again.

**Lemma 16.** *In the setting of Lemma 15, if  $\min\{p_0, 1 - p_0\} \leq \rho$ , the marginal probability will reach the closer border from  $\{1/n, 1 - 1/n\}$  in  $O((\log n)/\rho)$  steps with probability at least  $e^{-2/(1-e)}$ . This even holds if each step is a  $b$ -step.*

**Proof.** We consider only the case  $X_0 \leq \rho$  as the other case is symmetrical. The idea is to consider  $O(\log n)$  phases and prove that the  $X_t$ -value only decreases throughout all phases with the stated probability. Phase  $i$ , where  $i \geq 0$ , starts at the first time where  $X_t \leq \rho e^{-i}$ . Clearly, as  $\rho \leq 1$ , at the latest in phase  $\ln n$  the border  $1/n$  has been reached. We note that phase  $i$  ends after  $1/\rho$  steps if all these steps decrease the value; here we use that each step decreases by a relative amount of  $1 - \rho$  and that  $(1 - \rho)^{1/\rho} \leq e^{-1}$ .

The probability of decreasing the  $X_t$ -value in a step of phase  $i$  is at least  $(1 - \rho e^{-i})^2 \geq 1 - 2e^{-i}\rho$  even if the step is a b-step. Hence, the probability of all steps of phase  $i$  being decreasing is at least  $(1 - 2e^{-i}\rho)^{1/\rho} \geq e^{-2e^{-i}}$ . For all phases together, the probability of only having decreasing steps is still at least

$$\prod_{i=0}^{\ln n} e^{-2e^{-i}} \geq e^{-2 \sum_{i=0}^{\infty} e^{-i}} = e^{-2/(1-e)}$$

as suggested.  $\square$

We have now collected all tools to prove the lower bound for 2-MMAS<sub>ib</sub>.

**Proof of Theorem 6.** This follows mostly the same structure as the proof of Theorem 5. Every occurrence of the update strength  $1/K$  should be replaced by  $\rho$ . The analysis of b-steps is the same.

There is a minor change in the analysis of rw-steps. The two applications of Lemma 7 are replaced with Lemma 15, followed by an additional application of Lemma 16. The slightly different constants in the statement of Lemma 7 do not affect the asymptotic bound  $\Omega(n^{-\beta})$  obtained. Neither does the additional application of Lemma 16, which gives a constant probability. We do not care about the time  $O((\log n)/\rho)$  stated in Lemma 16, since we are only interested in a lower bound on the hitting time. Still, the assumptions on b-steps in Lemma 15 differ slightly from the ones in Lemma 7. We have to verify these new assumptions.

Lemma 15 requires in its first statement that the probability of a b-step is at most  $\rho/(4\alpha)$ . Recall that such a step has probability  $O(1/\sqrt{n})$ . We argue that  $\rho/(4\alpha) \geq c/\sqrt{n}$  for any constant  $c > 0$  if  $\kappa$  is small enough. To see this, we simply recall that  $\alpha = \kappa\sqrt{n}\rho/(3s^2)$  by definition and  $|s| = \Omega(1)$ .

Finally, the second statement of Lemma 15 restricts the number of b-steps until time  $\alpha(s/\rho)^2$  to at most  $s/(2\alpha\rho)$ . Reusing that  $\rho = O(\alpha/(\kappa\sqrt{n}))$ , this holds by Chernoff bounds with high probability if  $\kappa$  is a sufficiently small constant. Hence, the application of the lemma is possible.  $\square$

## 6 Conclusions

We have performed a runtime analysis of two probabilistic model-building GAs, namely cGA and 2-MMAS<sub>ib</sub>, on ONEMAX. The expected runtime of these algorithms was analyzed in dependency of the so-called update strength  $S = 1/K$  and  $S = \rho$ , respectively, resulting in the upper bound  $O(\sqrt{n}/S)$  for  $S = O(1/\sqrt{n} \log n)$  and  $\Omega(\sqrt{n}/S + n \log n)$ . Hence,  $S \sim 1/\sqrt{n} \log n$  was identified as the choice for the update strength leading to asymptotically smallest expected runtime  $\Theta(n \log n)$ .

Our analyses of update strength reveal a general trade-off between the speed of learning and genetic drift. High update strengths imply globally a fast adaptation of the probabilistic model but impact the overall correctness of the model negatively, resulting in increased risk of adapting to samples that are locally incorrect. We think that this

constitutes a universal limitation of the algorithms that extends to more general classes of functions. As even on the simple ONEMAX the update strength should not be bigger than  $1/(\sqrt{n} \log n)$ , we propose this setting as a general rule of thumb.

Our analyses have developed a quite technical machinery for the analysis of genetic drift. These techniques are not necessarily limited to cGA and 2-MMAS<sub>ib</sub> on ONEMAX. We are optimistic to be able to extend them other EDAs such as the UMDA [3] and even classical GAs such as the simple GA [19], where currently only quite restricted lower bounds on the runtime are available.

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## A General Tools

### A.1 Generalized Variable Drift Theorem

The following theorem is an easy generalization of [20, Theorem 1].

**Theorem 17** (Generalized variable drift theorem). *Consider a stochastic process on  $\mathbb{N}_0$ . Suppose there is a monotonic increasing function  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that the function  $1/h(x)$  is integrable on  $[1, m]$ , and with*

$$\Delta_k \geq h(k)$$

*for all  $k \in \{1, \dots, m\}$ . Then the expected first hitting time of any state from  $\{0, \dots, a-1\}$  for  $a \in \mathbb{N}$  is at most*

$$\frac{a}{h(a)} + \int_a^m \frac{1}{h(x)} dx.$$

### A.2 Bounds on the cumulative distribution function of the standard normal distribution

To prove Lemmas 7 and 15, we need the following estimates for  $\Phi(x)$ . More precise formulas are available (and can be found by searching for bounds on the so-called error function), but are not required for our analysis.

**Lemma 18** ([9], p. 175). *For any  $x > 0$*

$$\left( \frac{1}{x} - \frac{1}{x^3} \right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq 1 - \Phi(x) \leq \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

*and for  $x < 0$*

$$\left( \frac{-1}{x} - \frac{-1}{x^3} \right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq \Phi(x) \leq \frac{-1}{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$